Instabilities in a Plasma-Beam System Immersed in a Magnetic Field

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The interaction of an electron or an ion beam moving with velocity βc through a stationary plasma in the presence of a static magnetic field is investigated theoretically under the assumption that the plasma is cold, the intensity of the beam is very small, and that the beam, as well as the waves resulting from the interaction, are aligned along the direction of the magnetic field. It is well known that an electron beam moving with velocity exceeding the phase velocity of electromagnetic waves in a stationary plasma is capable of exciting such waves and these waves (circularly polarized) rotate in the same direction and at the same angular frequency as the gyroelectrons of the beam. It is shown in this investigation that there is an apparent reversal in the direction of rotation of these gyroelectrons as seen by the stationary observer. Consequently, the excited wave has a circular motion in the same direction as perturbed stationary ions. The frequency $\bar{\omega}$ of the excited wave satisfies the inequality $|\bar{\omega}|<\Omega_i$, where Ω_i is the gyrofrequency of perturbed stationary ions. Similarly, a wave excited by an ion beam has a circular motion in the same direction as perturbed stationary electrons. The frequency $\bar{\omega}^i$ of the latter wave satisfies the inequality $|\bar{\omega}^i|<\Omega_e$, where Ω_e is the gyrofrequency of perturbed stationary electrons. It is shown that an electron (ion) beam moving in the direction of the magnetic field can excite only a wave having negative (positive) helicity. The reverse situation occurs for a beam moving against the magnetic field. General relationships are formulated and illustrated graphically for determining the frequency and growth rate of waves which can be excited by an electron or an ion beam in any magnetized cold plasma. A particular case is illustrated in which an incident ion beam interacts with a relative dense plasma such as occurring in thermonuclear discharges, ionosphere, and interstellar space. It is shown that in such cases there is an excitation of hydromagnetic waves for a wide range of velocities of the ion beam.

INTRODUCTION

THIS investigation deals with instabilities resulting from the interaction of a beam of charged particles with plasma. These interactions are generally classified as electrostatic and electromagnetic. The electromagnetic interactions result in two oscillatory modes. One of these is transverse, i.e., the electric field intensity **E** is perpendicular to the wave vector **k**. The other mode is hybrid,[1] i.e., the electric field intensity **E** has components which are parallel and components which are perpendicular to the wave vector **k**. In the absence of a static magnetic field, the transverse mode is stable and the hybrid mode is unstable.

When a plasma beam system is immersed in a static magnetic field having induction $\mathbf{B}_0$, the transverse mode is unstable. According to Dawson and Bernstein, and Bernstein and Trehan,[2] the instability produced by an electron beam occurs in the presence of a resonance between the cyclotron frequency of the electrons in the beam and the frequency of the circularly polarized wave. There exists also an analogy between this plasma-beam instability and the anomalous Doppler effect.[3] This analogy has been investigated by Zhelezniakov.[4] There is extensive literature dealing with transverse plasma-beam instabilities in the presence of a magnetic field. It includes contributions of Weibel,[5] Harris,[6] Kovner,[7] Stepanov and Kitzenko,[8] Dokuchaev,[9] Tzintzadze and Lominadze,[10] Ginzburg,[11] and a number of other investigations.

This investigation deals with a simple formulation of the plasma-beam problem, and such factors as temperature, close collisions, nonuniform density distribution, etc., are not taken into account. The plasma is cold, the intensity of the beam is very small, and the wave vector **k**, the beam velocity $\mathbf{V}=\boldsymbol{\beta}c$, and the magnetic induction $\mathbf{B}_0$ are assumed to be parallel one to the other. The term c designates the velocity of light. It is assumed that the beam and the stationary plasma are of uniform density and infinite in extent.

In this investigation relationships are formulated for determining the frequency and the rate of growth of transverse waves which may be excited by an electron or an ion beam. These relationships are expressed in both analytical and graphical form. Some of the general results have been tabulated and classified in accordance with a system introduced by Denisse and

* Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

[1] Jacob Neufeld and P. H. Doyle, Phys. Rev. **121**, 654 (1961).

[2] J. Dawson and I. B. Bernstein, Paper presented at the Controlled Thermonuclear Conference, Washington, D. C.; TID-7558, 360 (1958). I. B. Bernstein and K. Trehan, Nuclear Fusion **1**, 3 (1960).

[3] For a discussion on anomalous Doppler effect, see V. L. Ginzberg and I. M. Frank, Doklady Akad. Nauk. S.S.S.R. **56**, 583 (1947).

[4] V. V. Zhelezniakov, Izv. Vysshikh Uchebn. Zavedenii, Radiofiz. **2**, 14 (1959); **3**, 57 (1960).

[5] E. S. Weibel, Phys. Rev. Letters **2**, 83 (1959).

[6] E. G. Harris, Plasma Phys. **2**, 138 (1961).

[7] M. S. Kovner, Zh. Eksperim. i Teor. Fiz. **40**, 527 (1961) [translation: Soviet Phys.—JETP **13**, 369 (1961)].

[8] K. N. Stepanov and A. B. Kitzenko, Zh. Tekh. Fiz. **31**, 167 (1961) (translation: Soviet Phys.—Tech. Phys. **6**, 120 (1961)]; A. B. Kitzenko and K. N. Stepanov, *ibid.* **31**, 176 (1961) [translation: *ibid.* **6**, 127 (1961)].

[9] V. P. Dokuchaev, Zh. Eksperim. i Teor. Fiz. **39**, 413 (1961) [translation: Soviet Phys.—JETP **12**, 294 (1961)].

[10] N. L. Tzintzadze and D. G. Lominadze, Zh. Tekh. Fiz. **31**, 1039 (1961) [translation: Soviet Phys.—Tech. Phys. **6**, 759 (1962)].

[11] M. A. Gintsburg, Zh. Eksperim. i Teor. Fiz. **41**, 752 (1961) [translation: Soviet Phys.—JETP **14**, 542 (1962)]; Phys. Rev. Letters **7**, 399 (1961).

Delcroix.[12] It has been found that an ion beam of any velocity within a wide velocity range incident on a "dense plasma" (such as the ionosphere, interstellar clouds, and thermonuclear plasma) excites hydromagnetic waves. The term "hydromagnetic waves" is applied in this investigation to circularly polarized waves, whereas Alfvén waves designate plane polarized waves. Both hydromagnetic and Alfvén waves have frequency $\omega \ll \Omega_i$, where Ω_i is the ion gyrofrequency of the stationary plasma. These waves are propagated with Alfvén velocity $V_A=(B_0^2/4\pi n M_i)^{1/2}$, where n is the plasma density and M_i the mass of an ion.

There is a definite relationship between the direction of the magnetic field, the direction of the beam, the direction of propagation, and the sense of rotation of an unstable circularly polarized wave. This relationship is described in this investigation.

1. PASSAGE OF AN ELECTRON BEAM THROUGH PLASMA

I. Dispersion Equation

Consider a charge equilibrated system composed of electrons and singly charged ions. Assume that the ions and the fraction $(1-\sigma)$ of the electrons are at rest while the fraction σ of the electrons are moving with velocity $\mathbf{v}=\beta c$ in the direction of the external magnetic field $\mathbf{B}_0$. It is assumed that the beam is very weak, i.e., $\sigma \ll 1$. Let

$$\omega_i=(4\pi n e^2/M_i)^{1/2};\quad \omega_e=(4\pi n e^2/m)^{1/2}, \tag{1-1}$$

where n is the density of ions, M_i is the mass of an ion, and e and m are the charge and mass of an electron. Let the term

$$\Omega_i=|e|B_0/M_i c \tag{1-2}$$

represent the ion gyrofrequency and the term

$$\Omega_e=|e|B_0/mc \tag{1-3}$$

represent the electron gyrofrequency. It is assumed that the positive linear direction is the direction of $\mathbf{B}_0$.

The dispersion equation for circularly polarized transverse waves propagating along the external magnetic field $\mathbf{B}_0$ can be written as follows[13]:

$$\phi\equiv\phi(\omega,k)=F-\frac{\sigma(1-\beta^2)^{1/2}\omega_e^2(\omega-ck\beta)}{\omega-ck\beta-\Omega_e(1-\beta^2)^{1/2}}=0, \tag{1-4}$$

where, taking into account the inequality $\sigma \ll 1$,

$$F\equiv F(\omega,k)=\omega^2-c^2k^2-\frac{\omega_i^2\omega}{\omega+\Omega_i}-\frac{\omega_e^2\omega}{\omega-\Omega_e}. \tag{1-5}$$

The term $\omega^2-c^2k^2$ in (1-5) represents the contribution to the dispersion equation of Maxwell's equations in vacuum. The term $\omega_i^2\omega/(\omega+\Omega_i)$ results from the plasma ions, the term $\omega_e^2\omega/(\omega-\Omega_e)$ results from the plasma electrons, and the term

$$\sigma(1-\beta^2)^{1/2}\omega_e^2(\omega-ck\beta)/[\omega-ck\beta-\Omega_e(1-\beta^2)^{1/2}]$$

in (1-4) results from the beam.

In expressions (1-4) and (1-5) the quantity ω represents the angular frequency of a field vector associated with a circularly polarized wave, and k is the wave number. Both ω and k may be positive or negative. Thus ω is positive if a field vector such as the electric intensity rotates clockwise when the observer is looking in the positive direction (i.e., in the direction of the magnetic field). The phase velocity of the wave is represented by the quantity ω/k. The sign of this quantity indicates the direction of propagation of the wave, i.e., if $\omega/k>0$, the wave is propagated in the direction of $\mathbf{B}_0$ and if $\omega/k<0$ the wave is propagated in the direction opposite to that of $\mathbf{B}_0$.

A circularly polarized wave has a positive or a negative helicity. The term positive helicity designates a wave in which the electric vector rotates clockwise as the wave moves away from the observer. For such a wave designated as an H_+ wave, one has $\omega>0$ and $\omega/k>0$ or $\omega<0$ and $\omega/k<0$. On the other hand, for an H_- wave having negative helicity, one has $\omega>0$ and $\omega/k<0$ or $\omega<0$ and $\omega/k>0$. Hence the sign of the wave number k determines the helicity of the wave. For $k>0$ one has a wave of positive helicity or an H_+ wave, and for $k<0$ the wave has negative helicity or an H_- wave. The wave having positive helicity is often designated as "left-handed polarized wave."[14]

II. Solution of the Dispersion Equation

1. Characteristic Frequency and Rate of Growth

Consider the dispersion equation (1-4). The term F in this equation is independent of the parameters of the beam so that the equation $F=0$ represents the dispersion equation for waves in the stationary plasma.

A comparison will be made between the solutions of Eq. (1-4), where F is given in (1-5). Following the customary procedure, one solves these equations for ω assuming that k is real. The values of ω obtained from $F=0$ represent the frequencies of waves in a stationary plasma. Since the stationary plasma is transparent, the values of ω are real and are represented as

$$\omega=\omega', \tag{1-6}$$

[12] J. F. Denisse and J. L. Delcroix, *Theorie des Ondes dans les Plasmas* (Dunod, Paris, 1961). The quantity A used by Denisse and Delcroix is approximately the square of the quantity A defined in this investigation.

[13] See, for instance, reference 2. A similar dispersion equation for a multibeam system was formulated by V. A. Bailey, Phys. Rev. **83**, 439 (1951).

[14] It seems appropriate to identify an H_+ wave with the advance of a right-handed screw. There is some confusion in the existing literature. Thus the H_+ wave is sometimes designated as left-handed polarized wave. See, for instance, J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), p. 206; or J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 280; I. B. Bernstein and K. Trehan, Nucl. Fusion **1**, 3 (1960). The same wave is also designated as "right wave" or "right-handed polarized wave." See, for instance, V. N. Kessenikh, "Rasposlvanienie Radiovoln," GITTL, 1952, p. 228; or V. L. Ginzburg, *Propagation of Electromagnetic Waves in Plasma,* translated by Royer and Roger (Gordon and Breach, New York, N. Y., 1960), p. 180.

where ω' is real. The presence of the beam [i.e., the last term in Eq. (1-4)] produces a perturbation of the roots of the equation $F=0$. The roots of (1–4) can thus be expressed

$$\omega=\omega'+\delta', \tag{1-7}$$

where[15]

$$\lim_{\sigma\to 0}\delta'=0. \tag{1-8}$$

If δ' is complex, then Imδ, if positive, represents the growth rate of the excited wave. The value ω' in (1-6) shall be designated as the "characteristic frequency" and the term Reδ' as the frequency shift of the excited wave.

2. *Resonant and Nonresonant Waves*

In order to clarify some of the characteristic features of a plasma-beam instability, one may differentiate between waves that are "resonant" and waves that are "nonresonant" with the beam. The frequency of a resonant wave will be expressed as

$$\omega=\tilde{\omega}, \tag{1-9}$$

where

$$\tilde{\omega}\approx ck\beta+\Omega_e(1-\beta^2)^{1/2}. \tag{1-10}$$

The frequency of a nonresonant wave will be expressed as

$$\omega=W, \tag{1-11}$$

where

$$W\neq ck\beta+\Omega_e(1-\beta^2)^{1/2}. \tag{1-12}$$

The designation "resonant" and "nonresonant" is based on a resonance between the cyclotron frequency of the electrons in the beam and the frequency of the wave.[2] The existence of such a resonance can be ascertained by means of the following considerations.

The electrons moving with the beam, when perturbed by an external electromagnetic field, rotate in a plane perpendicular to the motion of the beam. The gyrofrequency as measured by an observer moving with the beam can be expressed as

$$\Omega_e=\frac{eB_0(1-\beta_t^2)^{1/2}}{mc}, \tag{1-13}$$

where β_t represents the peripheral velocity acquired by the electron as a result of the perturbation. Since the perturbation is small, one has $\beta_t\ll 1$, and, therefore, it is assumed that

$$\Omega_e=eB_0/mc. \tag{1-14}$$

By applying Lorentz transformation to (1-13) it can be shown that the electron gyrofrequency as seen by a stationary observer has a value $(\Omega_e)_{st}$ which may be expressed as

$$(\Omega_e)_{st}=ck\beta+\Omega_e(1-\beta^2)^{1/2}. \tag{1-15}$$

[15] See, for instance, A. I. Akhiezer and Ia. B. Fainberg, Zhur. Eksp. Teoret. Fiz. **21**, 1262 (1951).

A wave is resonant with the beam if the electromagnetic field rotates with the same angular frequency ω as the perturbed electron. Thus, using equality (1-10), one has

$$\omega=(\Omega_e)_{st}\equiv\tilde{\omega}. \tag{1-16}$$

The frequency of a wave which is resonant with the beam is determined by the properties of the beam and is always expressed by a real quantity. However, if the beam passes through a plasma, the resonant wave is perturbed. The frequency of such a perturbed wave may be expressed in the form

$$\omega=\tilde{\omega}+\delta, \tag{1-17}$$

where the term δ represents the perturbation produced by the plasma. If Im$\delta>0$, expression (1-16) represents an "excited resonant wave."

Similarly, the frequency of a perturbed nonresonant wave passing through a plasma may be expressed as follows:

$$\omega=W+\delta'', \tag{1-18}$$

where the term δ'' represents the perturbation produced by the plasma.

3. *Relationship between the Angular Frequency $\tilde{\omega}$ of a Resonant Wave and the Linear Velocity $c\beta$ of a Relativistic Beam*

A discussion will be given on the relationship $\tilde{\omega}=ck\beta+\Omega_e(1-\beta^2)^{1/2}$ between the frequency of a resonant wave and the velocity of the beam. The discussion will be confined to the case shown in Fig. 1 which covers an H_+ wave, i.e., k constant and positive. A similar discussion could be given for an H_- wave, i.e., for k constant and negative. In the latter case the corresponding graph would be obtained by reflecting the curve in Fig. 1 about the vertical axis. The beam velocities β, represented by abscissas, vary within the range from

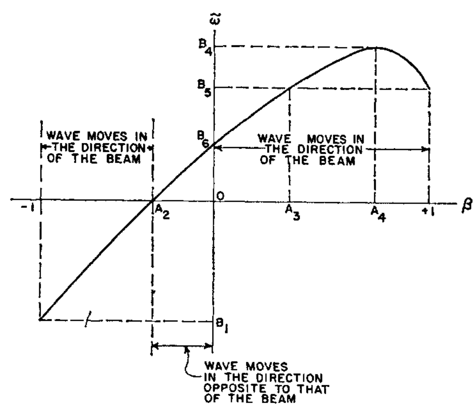

FIG. 1. Relationship between the angular frequency $\tilde{\omega}$ of a resonant wave and the linear velocity β of the electron beam. $OB_5=-OB_1=ck$; $OB_4=(c^2k^2+\Omega_e^2)^{1/2}$; $OA_4=ck/(c^2k^2+\Omega_e^2)^{1/2}$; $OA_2=-\Omega_e/ck$; $OB_6=\Omega_e$.

$\beta=-1$ to $\beta=+1$, and the corresponding values of $\tilde{\omega}$ are plotted as ordinates.

For a "stationary beam," i.e., when $\beta=0$, one has $\tilde{\omega}=OB_6=\Omega_e$. When the beam moves in the direction of the magnetic field, i.e., for $\beta>0$, the resonant frequency remains positive. When the direction of the beam is reversed, i.e., for $\beta<0$, the resonant frequency decreases and for

$$\beta=OA_2=-\Omega_e/ck, \tag{1-19}$$

one has $\tilde{\omega}=(\Omega_e)_{st}=0$. Furthermore, for

$$-1<\beta<-\Omega_e/ck, \tag{1-20}$$

the quantity $\tilde{\omega}$ is negative.

It will be shown subsequently that an electron beam may excite an H_+ or an H_- wave only if the following two conditions are satisfied: (1) the wave is resonant with the beam, and (2) the characteristic frequency of the wave is negative. Therefore, negative frequencies are of primary interest in this investigation. It can be seen from Fig. 1 that in the regions in which the instability may occur, i.e., for $\tilde{\omega}<0$, the resonant H_+ wave moves in the same direction as the beam. It can also be shown that for $k<0$ in the region for which $\tilde{\omega}<0$ the resonant H_- wave moves in the same direction as the beam.

4. *Subluminous and Superluminous Velocities*[16]

The phase velocity of a resonant wave is represented by a quantity

$$V_{\mathrm{ph}}=\tilde{\omega}/k. \tag{1-21}$$

Consider an expression R representing the ratio of the velocity $c\beta$ of the beam to the phase velocity V_{ph} of a wave resonant with the beam. One has

$$R=c\beta/V_{\mathrm{ph}}=ck\beta/\tilde{\omega}. \tag{1-22}$$

Using (1-15) and (1-16), one obtains

$$R=1-\frac{\Omega_e(1-\beta^2)^{1/2}}{\tilde{\omega}}. \tag{1-23}$$

Expression (1-23) represents the relationship between R and $\tilde{\omega}$ for a fixed value of β. This relationship is illustrated graphically in Fig. 2. [The scales in all figures except Figs. 6, 7, 13(a), 13(b), and 14 are considerably distorted in order to show the qualitative features of the graphs.]

Consider separately in the above representation positive frequencies ($\tilde{\omega}>0$) and negative frequencies ($\tilde{\omega}<0$).

[16] Some time ago it was found expedient to introduce into the scientific language such terms as "subsonic" and "supersonic" velocities in order to designate velocities which are, respectively, lower and higher than the velocities of sound. There appears to be a need for a similar terminology to designate velocities that are respectively lower and higher than the velocity of an electromagnetic wave moving in the same direction as the wave in a given medium. It is hoped that the suggested terms "subluminous velocities" and "superluminous velocities" will be accepted by others.

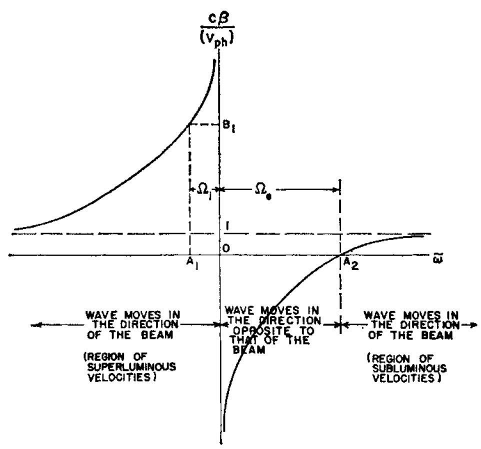

FIG. 2. Relationship between the beam velocity $c\beta$ and the phase velocity (V_{ph}) of a wave resonant with the beam. $OB_1=[\alpha+(1-\beta^2)^{1/2}]/\alpha$; $OB_2=1$.

For values of $\tilde{\omega}$ satisfying the inequality $\tilde{\omega}>\Omega_e$, one obtains from (1-23)

$$0<R=c\beta/V_{\mathrm{ph}}<1. \tag{1-24}$$

Consequently, the velocity of the beam is "subluminous." The beam moves in the same direction as the wave, and its velocity is lower than that of the wave.

For frequencies satisfying the inequality $0<\tilde{\omega}<\Omega_e$, one obtains from (1-23)

$$R=c\beta/V_{\mathrm{ph}}<0. \tag{1-25}$$

Consequently, the beam moves in the direction opposite to that of the wave.

Of particular interest is the case for which $\tilde{\omega}$ is negative since it is only in this case that an instability may occur. For $\tilde{\omega}<0$ one has

$$R=c\beta/V_{\mathrm{ph}}>1. \tag{1-26}$$

Consequently, the velocity of the beam is "superluminous." The beam moves in the same direction as the wave and its velocity is higher than that of the wave.

5. *"Normal" and "Anomalous" Angular Velocities*

Consider a stationary plasma immersed in a static magnetic field and perturbed by an incident electromagnetic wave. The electrons and ions which were initially stationary acquire, as a result of the perturbation, circular motions in the plane perpendicular to $\mathbf{B}_0$. If the magnetic field is directed away from the observer, then the electrons turn clockwise with the angular frequency Ω_e, and the ions turn counterclockwise with the angular frequency Ω_i.[17] These rotational velocities are designated as "normal." Thus Fig. 3(a) shows the "normal" circular motion of a perturbed electron and a

[17] See, for instance, H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1953).

perturbed ion. This figure also shows the circular motion of a wave in resonance with the perturbed electron. Such a wave may be either an H_+ wave propagating away from the observer or an H_- wave propagating towards the observer. It is assumed in Fig. 3(a) and also in Figs. 3(b) and 3(c) that the resonant wave is an H_+ wave. The discussion will be limited to this case.

Assume that an electron beam aligned along the magnetic field moves with a velocity βc and consider the circular motion which an electron in such a beam acquires as a result of perturbation. If a beam moves with subluminous velocity, the rotational velocity of perturbed electrons in the beam is normal. However, when the beam is superluminous, there is a reversal in the rotational velocity in the framework of a stationary observer. Such a reversal may be observed by comparing Figs. 3(b) and 3(c). Figure 3(b) illustrates the case for which β satisfies the inequality

$$-\Omega_e/ck<\beta<1, \tag{1-27}$$

and Fig. 3(c) illustrates the case for which β satisfies the inequality

$$-1<\beta<-\Omega_e/ck. \tag{1-28}$$

Consider now Fig. 3(b). In the range $0<\beta<1$ the beam moves in the direction of the magnetic field with subluminous velocity. In the range for which $-\Omega_e/ck<\beta<0$, the direction of the beam is opposite to that of the wave. This direction is also opposite to that of the magnetic field. Both the perturbed electron and the wave have "normal" clockwise motions similar to those shown in Fig. 3(a).

Consider now Fig. 3(c). For values of β satisfying inequality (1-28) the beam moves with superluminous velocity in the direction which is opposite to that of the magnetic field. One obtains here an apparently paradoxical situation in which the angular velocity of an electron moving with the beam as seen by the stationary observer is reversed. This "anomalous rotational velocity" is shown in Fig. 3(c). Both the perturbed electron and the electromagnetic field, as seen by the stationary observer, rotate counterclockwise. Therefore, the H_+ wave rotates in the same direction as the stationary ion and moves in the upward direction against the magnetic field.

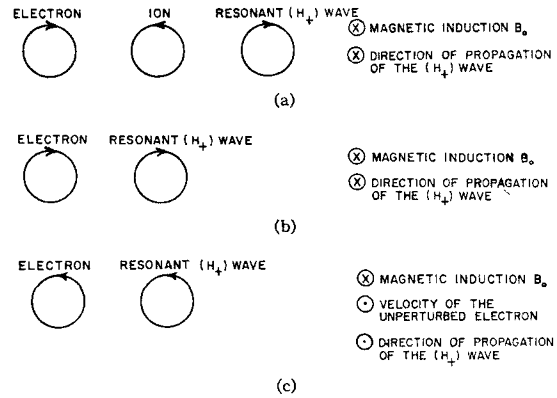

FIG. 3. (a) Circular motion of a perturbed electron, perturbed ion, and of a resonant H_+ wave. In the absence of the perturbation, the electron and the ion are at rest. Both the magnetic induction $\mathbf{B}_0$ and the H_+ wave are directed downward through the paper. (b) Circular motion of a perturbed electron and a resonant H_+ wave. In the absence of the perturbation the electron moves with subluminous velocity. Both the magnetic induction B_0 and the H_+ wave are directed downward through the paper. (c) Circular motion of a perturbed electron and a resonant H_+ wave. In the absence of the perturbation, the electron moves with superluminous velocity. The magnetic induction $\mathbf{B}_0$ is directed downward through the paper. The H_+ wave and the electron, when unperturbed, are directed upward through the paper.

6. *Criterion for an Instability*

In describing a plasma-beam instability, one may differentiate between the effects which are directly dependent on the character of the stationary plasma and the effects dependent on the characteristics of the beam.

Waves in a stationary plasma are characterized by the quantities ω and k which satisfy the dispersion equation $F(\omega,k)=0$. These waves may be resonant with the beam [i.e., satisfy the relationship $\omega=\tilde{\omega}$ where $\tilde{\omega}$ is given by (1-10)] or nonresonant with the beam [i.e., satisfy the relationship $\omega=W$ where W satisfies (1-12)].

In the first-order approximation the nonresonant waves cannot be excited by the beam. This can be seen by substituting $\omega=W+\delta''$, where $\delta''\to 0$ as $\sigma\to 0$, into the dispersion equation (1-4). Assuming $|\delta''|\ll|W|$, $|\delta''|\ll|W-ck\beta-\Omega_e(1-\beta^2)^{1/2}|$, and approximating F in the neighborhood of $\omega=W$ by a Taylor series, the solution

$$\delta''=-\frac{(F)_{\omega=W}[W-ck\beta-\Omega_e(1-\beta^2)^{1/2}]-\sigma\omega_e^2(1-\beta^2)^{1/2}(W-ck\beta)}{(\partial F/\partial\omega)_{\omega=W}[W-ck\beta-\Omega_e(1-\beta^2)^{1/2}]-\sigma\omega_e^2(1-\beta^2)^{1/2}} \tag{1-29}$$

can be obtained which is seen to be real. Consequently, the nonresonant waves neither grow nor decay and are of no interest in this investigation.

Consider a resonant wave in a plasma-beam system. The frequency of this wave is $\omega=\tilde{\omega}+\delta$. Furthermore, it is assumed that for sufficiently small values of σ one has $|\delta|\ll|\tilde{\omega}|$. In order to determine whether a resonant wave may be excited by a beam, one needs to substitute $\omega=\tilde{\omega}+\delta$ in Eq. (1-4) and determine whether or not this equation gives complex roots for δ having $\mathrm{Im}\delta>0$.

Approximating F in the neighborhood of $\omega=\tilde{\omega}$ by Taylor series and retaining the first two terms in the series, one obtains

$$(F)_{\omega=\tilde{\omega}+\delta}=(F)_{\omega=\tilde{\omega}}+\delta(\partial F/\partial\omega)_{\omega=\tilde{\omega}}. \tag{1-30}$$

Applying the above expression and $\omega=\tilde{\omega}+\delta$ to the dispersion equation (1-4), one obtains

$$\delta^2(\partial F/\partial\omega)_{\omega=\tilde{\omega}}+\delta(F)_{\omega=\tilde{\omega}}-\sigma\omega_e^2\Omega_e(1-\beta^2)=0. \quad (1\text{-}31)$$

The discriminant of (1-31) is

$$\Delta=[(F)_{\omega=\tilde{\omega}}]^2+4\sigma\omega_e^2(1-\beta^2)(\partial F/\partial\omega)_{\omega=\tilde{\omega}}. \quad (1\text{-}32)$$

Expression (1-32) is positive if

$$(\partial F/\partial\omega)_{\omega=\tilde{\omega}}>0. \quad (1\text{-}33)$$

Consequently, when the inequality (1-33) is satisfied, there is no instability.

Consider now the inequality

$$(\partial F/\partial\omega)_{\omega=\tilde{\omega}}<0. \quad (1\text{-}34)$$

Expression (1-32) is negative provided one has the inequality (1-34) and the additional inequality

$$\left|\frac{[(F)_{\omega=\tilde{\omega}}]^2}{(\partial F/\partial\omega)_{\omega=\tilde{\omega}}}\right|<4\sigma\omega_e^2\Omega_e(1-\beta^2). \quad (1\text{-}35)$$

If σ is very small, the inequality (1-35) is satisfied only when

$$(F)_{\omega=\tilde{\omega}}\sim 0. \quad (1\text{-}36)$$

Therefore, there is always an instability when inequality (1-34) and condition (1-36) are satisfied. The expression (1-36) shows that the frequency has to be in the immediate neighborhood of the roots of the equation $F(\omega,k)=0$.

Taking into account the relationship (1-36), the following expression is obtained from (1-31):

$$\delta^2=\frac{\sigma\omega_e^2\Omega_e(1-\beta^2)}{(\partial F/\partial\omega)_{\omega=\tilde{\omega}}}. \quad (1\text{-}37)$$

Thus the rate of growth, Imδ, may be expressed as

$$\mathrm{Im}\delta=\left|\frac{\sigma\omega_e^2\Omega_e(1-\beta^2)}{(\partial F/\partial\omega)_{\omega=\tilde{\omega}}}\right|^{1/2}. \quad (1\text{-}38)$$

The above derivations are based on the assumption that $|\delta|\ll|\tilde{\omega}|$. Therefore, the values of $\tilde{\omega}$ which satisfy $\tilde{\omega}\sim 0$ should be excluded.

The behavior of an excited resonant wave may also be characterized by a nondimensional parameter

$$N=\mathrm{Im}\delta/\tilde{\omega}. \quad (1\text{-}39)$$

This parameter represents the relative rate of growth of the wave expressed in decibels per cycle. The quantity σ has to be sufficiently small so as to satisfy the inequality

$$N\ll 1. \quad (1\text{-}40)$$

7. *Graphical Representation of Excited Resonant Waves*

A graphical representation of plasma-beam instabilities is given in Figs. 4 and 5. Two different assumptions are made.

Figure 4 is based on an assumption that k is given. The problem consists in determining the frequencies $\tilde{\omega}$ of waves which may be excited by beams having various velocities β. To each value of β corresponds one or more values of $\tilde{\omega}$.

Figure 5 is based on an assumption that the velocity β of the beam is given. The problem consists in finding waves which may be excited by such a beam. Each of these waves is characterized by definite values $\tilde{\omega}$ and k.

A. Excited waves characterized by a given value of k. Consider a function

$$y=(F)_{\omega=\tilde{\omega}}, \quad (1\text{-}41)$$

which is illustrated graphically in Fig. 4 under the assumption that k is fixed. The zeros of this function satisfy the equation $(F)_{\omega=\tilde{\omega}}=0$ and are labeled A_1, A_3, A_4, and A_6.

It is noted that the values for $\tilde{\omega}$ in Fig. 4 extend from $-\infty$ to $+\infty$. However, not all of these values are physically significant since $\tilde{\omega}$ is bounded and comprised within the range $-ck<\tilde{\omega}\leq(c^2k^2+\Omega_e^2)^{1/2}$. Therefore, some portions of the graph of Fig. 4 are not applicable to the present problem, and some of the zeros of the function (1-41) should be disregarded.

It can be shown by substituting the maximum and minimum values for $\tilde{\omega}$ obtained from Fig. 1 into (1-41) and comparing the results with Fig. 4 that the root labeled A_1 must be disregarded, and the only negative root which has physical significance is labeled A_3. Similarly, it can be shown that under some conditions there are two positive roots labeled A_4 and A_6 which are physically significant. It can be shown that in such a case two different values of β yield the same frequency A_6. Under other conditions there is only one positive root labeled as A_4 which is physically significant, i.e., the point A_6 has to be discarded.

Thus, for a given value of k there are two (or three) values of $\tilde{\omega}$ representing the frequencies of resonant waves which may be transmitted through the stationary plasma. One of these frequencies, labeled by the point A_3, is negative while the remaining two are positive. In order to ascertain which of these waves may be excited, one needs to apply the criterion (1-34). Since the slope of the curve shown in Fig. 4 is positive at points A_4 and A_6, there is no instability for waves having frequencies labeled by these points. However, the slope is negative at the point A_3 and, therefore, there is an instability represented by this point.

The curve in Fig. 4 applies to either $k>0$ or $k<0$ and therefore is applicable to either H_+ waves or H_- waves. Thus it is seen that the instability occurs for negative values of $\tilde{\omega}$ for either an H_+ or H_- wave. It was pointed out in the discussion of Fig. 2 that negative frequencies correspond to a superluminous beam. Consequently, an instability occurs only when the beam has superluminous velocity. Referring now to Fig. 3, it is seen that there is a reversal in the direction of rotation of a wave which is in resonance with an electron beam

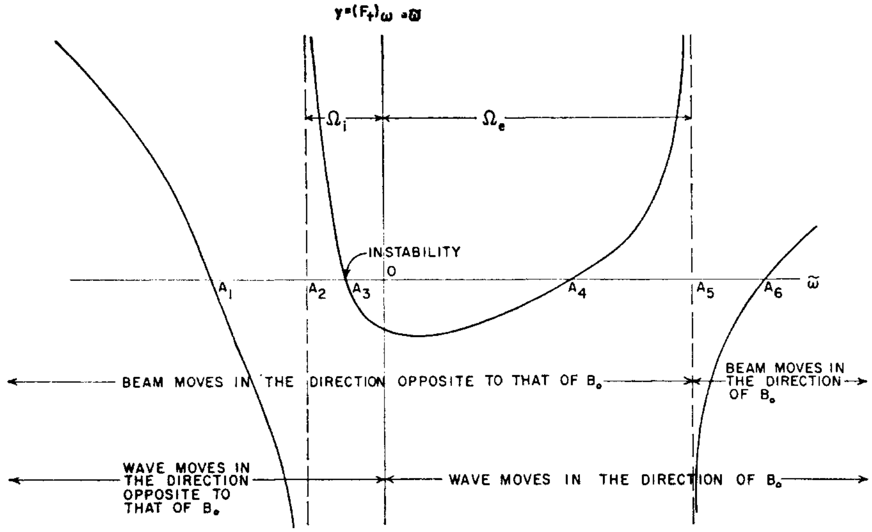

FIG. 4. Graphical representation of $y=(F)_{\omega=\tilde{\omega}}$ for a fixed value of k.

having superluminous velocity. This is in agreement with Fig. 4 which also shows that at resonance the frequency of the wave satisfies an inequality

$$-\Omega_i<\tilde{\omega}<0. \tag{1-42}$$

Consequently, the magnitude of the frequency of the excited wave is always below the ion gyrofrequency Ω_i. Thus one cannot obtain a resonance between an excited wave and the ion gyrofrequency. However, one can achieve a condition near resonance, i.e., the wave frequency may approach the ion gyrofrequency from below and the difference $\Omega_i-\tilde{\omega}$ may be relatively small. One can also have a relationship $|\tilde{\omega}|\ll\Omega_i$. It is shown in Sec. 3 of this investigation that, in such case, the electron beam excites a hydromagnetic wave.

The abscissas in Fig. 4 may represent either the values of $\tilde{\omega}$ or the corresponding values of β. The variables $\tilde{\omega}$ and β are related by the equality $\tilde{\omega}=ck\beta+\Omega_e(1-\beta^2)^{1/2}$ in which k is fixed. Thus, when $\beta=0$, one obtains $\tilde{\omega}=\Omega_e$ which is represented by the point A_5. The regions in which the beam velocity is in the direction of the magnetic field and against this direction are correspondingly marked in Fig. 4 for an H_+ wave ($k>0$). For an H_- wave ($k<0$), the directions of the beam and wave would be the reverse of those shown. It is noted that the instability occurs only when the wave and the beam have the same direction (opposite the direction of $\mathbf{B}_0$ for an H_+ wave and in the direction of $\mathbf{B}_0$ for an H_- wave).

B. Waves excited by a beam moving with velocity β. In various practical applications the velocity $c\beta$ of the beam is given. Therefore, in exploring the instabilities, one assumes that $\tilde{\omega}$ varies with k in accordance with the expression $\tilde{\omega}=ck\beta+\Omega_e(1-\beta^2)^{1/2}$ while β remains fixed. Consider now expression (1-5). Assuming that

$$\omega=\tilde{\omega},\quad k=[\tilde{\omega}-\Omega_e(1-\beta^2)^{1/2}]/c\beta, \tag{1-43}$$

and substituting the above values of ω and k, one obtains

$$(F)_{\omega=\tilde{\omega},\,k=[\tilde{\omega}-\Omega_e(1-\beta^2)^{1/2}]/c\beta}\equiv\psi(\tilde{\omega})$$
$$=\tilde{\omega}^2-\frac{1}{\beta^2}\left(\tilde{\omega}-\frac{\Omega_i(1-\beta^2)^{1/2}}{\alpha}\right)^2-\frac{\omega_i^2\tilde{\omega}^2(1+\alpha)}{(\tilde{\omega}+\Omega_i)(\alpha\tilde{\omega}-\Omega_i)}, \tag{1-44}$$

where $\alpha=m/M_i$.

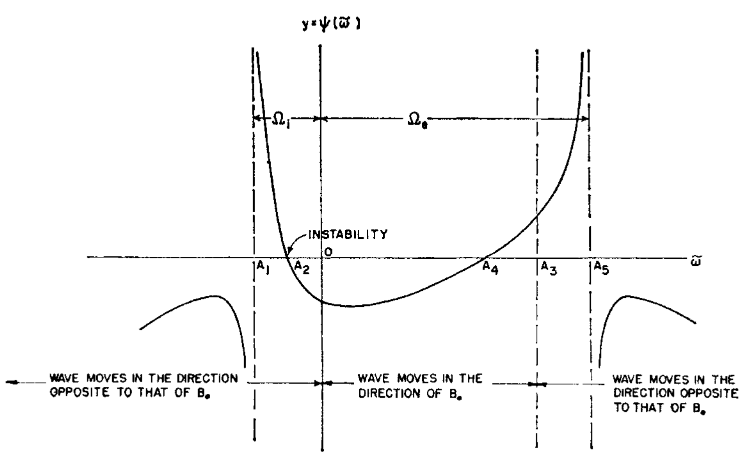

FIG. 5. Graphical representation of $y=\psi(\tilde{\omega})$ for a fixed value of β. $OA_3=\Omega_e(1-\beta^2)^{1/2}$.

Depending on the values of the parameters of the system, one may obtain a graph of the function $y=\psi(\tilde{\omega})$ as given in Fig. 5 or a similar graph (not shown) where the branch to the right of the point A_5 intersects the axis of abscissas. Thus there are two or four points of intersection of the graph $y=\psi(\tilde{\omega})$ with the axis of abscissas. The graph of Fig. 5 shows two such points which are labeled as A_2 and A_4. In order to determine which of these points represents an instability, one should consider the expression (1-37) for δ and ascertain whether this quantity is real or complex. Using (1-5) and (1-37), one can represent δ^2 as follows:

$$\delta^2=\frac{\sigma\omega_i^2\Omega_i(1-\beta^2)}{\alpha^2\tilde{\omega}[2+\omega_i^2\Omega_i(\tilde{\omega}+2\Omega_i)/(\tilde{\omega}+\Omega_i)^2(\alpha\tilde{\omega}-\Omega_i)^2]}. \quad (1\text{-}45)$$

It can be seen that no positive value of $\tilde{\omega}$ will make the expression (1-45) negative and, therefore, no positive solution for $\tilde{\omega}$ can give rise to an instability. On the other hand, any negative value of $\tilde{\omega}$ such that

$$|\tilde{\omega}|<2\Omega_i \quad (1\text{-}46)$$

will correspond to an instability. Therefore, there is an instability for any solution for $\tilde{\omega}$ satisfying the inequality $-\Omega_i<\tilde{\omega}<0$.

Figure 5 shows the frequency ranges for which the resonant wave moves with or against the magnetic field under the assumption that $\beta<0$, i.e., the beam is directed against the magnetic field.

8. Quantitative Study of Instabilities

A. Anomalous Doppler effect. A plasma-beam instability occurs when the beam moves with superluminous velocity. There is a definite relationship between the instability produced by a beam and the behavior of a single particle moving with superluminous velocity. In the case of a thermal plasma with no external magnetic field, a formal analogy has been established between the longitudinal instability produced by the beam and the longitudinal Vavilov-Čerenkov effect produced by a particle moving with the same velocity as the beam.[18] A similar analogy exists when the plasma-beam system is immersed in a static magnetic field. In such case the beam effect is expressed by a transverse instability and the particle effect by the anomalous Doppler radiation.[4]

The instability of the "Doppler wave" has been accounted for by Zhelezniakov[4] by the quantum theory of radiation. The perturbed electrons moving with the beam perform oscillatory motions at electron gyrofrequency, and, therefore, these electrons may be considered as oscillators having proper frequency $\omega_0=\Omega_e$. For an electron beam moving with subluminous velocity, the "Doppler radiation" emitted by any single electron will be absorbed by the beam itself. On the other hand, for a beam moving with superluminous velocity the "Doppler radiation" emitted by any single electron will be amplified by the emission of a photon and will induce other electrons to effect transitions into higher energy states. Consequently, there is a conversion of the kinetic energy of the beam into the energy of a growing electromagnetic wave.

The instability represented by the point A_3 in Fig. 4 or by the point A_2 in Fig. 5 is characterized by a negative frequency $\tilde{\omega}$. On the other hand, in the conventional representation of the Doppler effect (both normal and anomalous) the frequency radiated by the moving oscillator is always positive. Nevertheless, there is formal analogy between the representation in Figs. 4 and 5 and the conventional representation.

In order to point out the above analogy, consider the conventional representation of the Doppler effect. Assume that an oscillator having proper frequency ω_0 moves with subluminous velocity β in a dispersive medium characterized by a refraction coefficient n_r. One has then

$$\beta n_r<1, \quad (1\text{-}47)$$

and the frequency ω radiated by the oscillator along the direction of its motion is expressed as

$$\omega=\omega_0(1-\beta^2)^{1/2}/(1-\beta n_r). \quad (1\text{-}48)$$

Expression (1-48) represents the normal Doppler effect. Assume now that the velocity of the oscillator is superluminous. Then

$$\beta n_r>1, \quad (1\text{-}49)$$

and the frequency radiated by the oscillator along the direction of its motion is expressed as

$$\omega=-\omega_0(1-\beta^2)^{1/2}/(1-\beta n_r). \quad (1\text{-}50)$$

Expression (1-50) represents the anomalous Doppler effect.

Equations (1-48) and (1-50) are generally derived from the microscopic description of the emission of a photon by a moving oscillator. Assume that $\hbar\omega$ is the energy of the emitted photon, ΔE is the change in the kinetic energy of the oscillator which results from the emission, and $\hbar\omega_0$ is the change in the internal energy of the oscillator. One has

$$\Delta E=\hbar\omega\pm\hbar\omega_0. \quad (1\text{-}51)$$

In the normal Doppler effect, as expressed by (1-48), the oscillator effects a transition from a higher energy state to a lower energy state. In such case the energy $\hbar\omega_0$ of the emitted photon is compensated at least partly by the excitation energy $\hbar\omega_0$ of the oscillator. Thus the sign "minus" must be assigned in (1-51) to the term $\hbar\omega_0$. By combining (1-51) with the equation for the conservation of momentum, one obtains the expression (1-48). On the other hand, in case of an anomalous Doppler effect, the kinetic energy of the moving oscillator is converted into the energy $\hbar\omega$ of the emitted photon and into the excitation energy $\hbar\omega_0$. Thus by

[18] Jacob Neufeld, Phys. Rev. **127**, 346 (1962).

assigning "plus" to the term $\hbar\omega_0$, and combining (1-51) with the equation for the conservation of momentum, the expression (1-50) is obtained.

Consider now the formulation of radiative effects contained in this investigation and particularly expression (1-17) representing the frequency of a resonant wave in a plasma-beam system. It is assumed that σ is sufficiently small so that the inequality $|\delta|\ll|\tilde{\omega}|$ is satisfied. Let

$$n_r = ck/\tilde{\omega} \tag{1-52}$$

represent the index of refraction of the plasma-beam medium. Substituting (1-52) in (1-17) and assuming $\delta=0$, one obtains

$$\tilde{\omega}=\Omega_e(1-\beta^2)^{1/2}/(1-n_r\beta). \tag{1-53}$$

The above expression has a formal resemblance to expressions (1-48) and (1-50) representing the normal and anomalous Doppler effect associated with an oscillator having proper frequency ω_0 equal to the electron gyrofrequency Ω_e. It should be pointed out, however, that Eq. (1-53) is based on classical electromagnetic theory, i.e., no consideration was given to the microscopic behavior of a single oscillator. Thus, the quantity Ω_e is always positive and there is no reversal in the sign of Ω_e when subluminous velocity is replaced by superluminous velocity. There is, however, a reversal in the sign of the term $\tilde{\omega}$. The quantity $\tilde{\omega}$ is positive for subluminous velocities and negative for superluminous velocities. The considerations leading to this reversal of sign for superluminous velocities have been set forth in connection with Fig. 3(a–c).

B. Velocity range for beams which are effective in exciting electromagnetic waves. When k is fixed, excitation is obtained when the beam velocities are contained within a determined range. The extent of this range will now be established. Taking into account the inequality $-\Omega_i<\tilde{\omega}<0$ representing the range of frequencies that may be excited, and the relationship $\tilde{\omega}=ck\beta+\Omega_e(1-\beta^2)^{1/2}$, it can be ascertained that for $k>0$ the velocity β of a beam that causes the excitation is comprised within the range

$$\beta_a<\beta<\beta_b, \tag{1-54}$$

where

$$\beta_a=-\Omega_e/(\Omega_e^2+c^2k^2) \times\{ck\alpha+[c^2k^2+\Omega_e^2(1-\alpha^2)]^{1/2}\}, \tag{1-55}$$

and

$$\beta_b=-\Omega_e/(\Omega_e^2+c^2k^2)^{1/2}. \tag{1-56}$$

Assuming that $\alpha^2\ll 1$, (1-55) becomes

$$\beta_a=-[\Omega_e/(\Omega_e^2+c^2k^2)^{1/2}]-[ck\Omega_e\alpha/(\Omega_e^2+c^2k^2)]. \tag{1-57}$$

C. Frequencies of excited waves. Consider the equation $\psi(\tilde{\omega})=0$ where $\psi(\tilde{\omega})$ is given by (1-44). This equation can be represented as

$$X^2-\frac{1}{\beta^2}\left(X-\frac{(1-\beta^2)^{1/2}}{\alpha}\right)^2-\frac{A^2X^2(1+\alpha)}{(X+1)(\alpha X-1)}=0, \tag{1-58}$$

where

$$A=\omega_i/\Omega_i \quad\text{and}\quad X=\tilde{\omega}/\Omega_i. \tag{1-59}$$

It can be seen from Eq. (1-58) that for $X<0$ (i.e., when the velocity of the beam is superluminous), the equality (1-58) can be satisfied only if

$$-1<X<0. \tag{1-60}$$

This inequality is identical to the inequality (1-42) resulting from the graphical methods of Figs. 4 and 5.

Expression (1-58) represents a functional relationship between three nondimensional quantities: X, A, and β. This relationship specifies conditions under which an instability may occur. Thus, for a given stationary plasma (A and ω_i are known), one can determine the frequencies $\tilde{\omega}$ that may be excited by beams having various velocities β. Similarly, if the velocity β of the beam is known, one can ascertain the values of A for which the excitation may occur and the corresponding values of X. Various graphs representing the behavior of the relationship (1-58) are given in Fig. 6.

Taking into account $\alpha\ll 1$, the terms α or αX may be neglected when either is added to 1 and thus (1-58) can be represented in the following form:

$$\left[X^2(1-\beta^2)-\frac{2X(1-\beta^2)^{1/2}}{\alpha}+\frac{(1-\beta^2)}{\alpha^2}\right](X+1) -A^2\beta^2X^2=0. \tag{1-61}$$

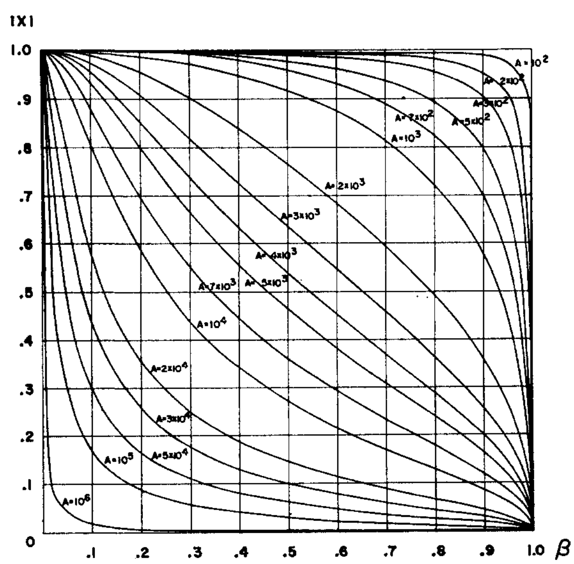

FIG. 6. Graphical representation of the relationship between X, A, and β (for an electron beam).

Since, $\alpha \ll 1$ and $-1<X<0$, it is clear that

$$|X^2(1-\beta^2)| \ll (1-\beta^2)/\alpha^2. \quad (1\text{-}62)$$

Therefore, taking into account (1-62), the expression (1-61) can be represented in the form

$$A^2=\frac{[1-\beta^2-2X\alpha(1-\beta^2)^{1/2}](X+1)}{\alpha^2\beta^2X^2}. \quad (1\text{-}63)$$

Using the equality (1-61), the quantity X can be expressed explicitly as a function of the parameters A and β of the plasma-beam system.

Using (1-62), Eq. (1-61) becomes

$$X^2\left[\frac{2(1-\beta^2)^{1/2}}{\alpha}+A^2\beta^2\right]+X\left[\frac{2(1-\beta^2)^{1/2}}{\alpha}-\frac{1-\beta^2}{\alpha^2}\right]-\frac{1-\beta^2}{\alpha^2}=0. \quad (1\text{-}64)$$

The solution of (1-64) can be expressed in the following form:

$$X=\frac{(1-\beta^2)^{1/2}/2\alpha-1\pm[(1+(1-\beta^2)^{1/2}/2\alpha)^2+A^2\beta^2]^{1/2}}{2+A^2\beta^2\alpha^2/(1-\beta^2)^{1/2}}. \quad (1\text{-}65)$$

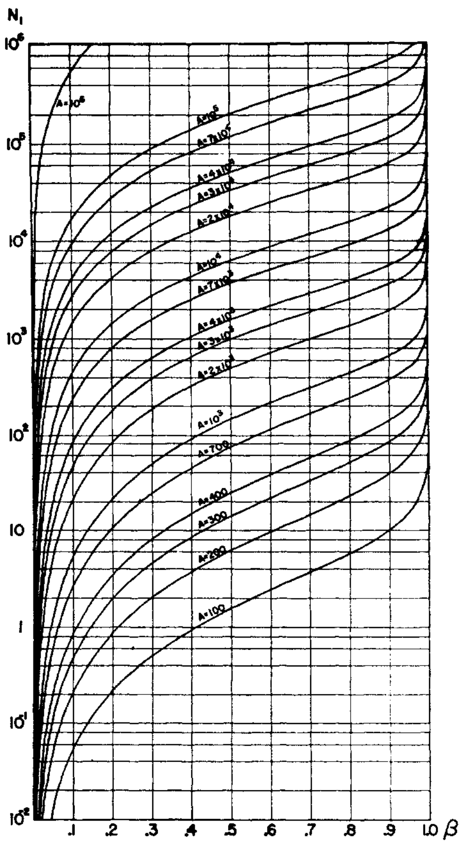

FIG. 7. Graphical representation of the relationship between N_1, A, and β (for an electron beam).

The only values of X which can yield an instability satisfy inequality (1-60) and, therefore, the solution corresponding to the $(+)$ sign in front of the radical should be discarded.

D. Rate of growth. The rate of growth $\mathrm{Im}\delta$ is obtained from (1-37) or (1-45). Using the nondimensional quantities X and A as given by (1-59), the growth rate can be represented as

$$\mathrm{Im}\delta=\left|\frac{\sigma\omega_i^2(1-\beta^2)(X+1)^2}{\alpha^2X[2(X+1)^2+A^2(X+2)]}\right|^{1/2}. \quad (1\text{-}66)$$

In deriving the above expression, the terms α and αX have been neglected when added to or subtracted from 1.

Consider now the relative rate of growth N represented by (1-39). This quantity can be expressed as

$$N=\sigma^{1/2}N_1, \quad (1\text{-}67)$$

where

$$N_1=\left|\frac{A^2(1-\beta^2)(X+1)^2}{\alpha^2X^3[2(X+1)^2+A^2(X+2)]}\right|^{1/2}. \quad (1\text{-}68)$$

Taking into account the inequality $N\ll 1$, one obtains

$$\sigma^{1/2}\ll 1/N_1. \quad (1\text{-}69)$$

Expression (1-69) indicates the restriction which the inequality (1-40) places upon the permissible density of the beam.

Substituting X as given by (1-65) into (1-68), one obtains

$$N_1=\left|\frac{(1-\beta^2)A^2T^3(U+T-S)^2}{\alpha^2(U-S)^3[2(U+T-S)^2+A^2T(U+2T-S)]}\right|^{1/2}, \quad (1\text{-}70)$$

where

$$T=4\alpha(1-\beta^2)^{1/2}+2\alpha^2A^2\beta^2, \quad (1\text{-}71)$$

$$S=2\alpha(1-\beta^2)^{1/2}\left[\left(1-\frac{(1-\beta^2)^{1/2}}{2\alpha}\right)^2+A^2\beta^2\right]^{1/2}, \quad (1\text{-}72)$$

and

$$U=1-\beta^2-2\alpha(1-\beta^2)^{1/2}. \quad (1\text{-}73)$$

Expression (1-70) describes a functional relationship between three nondimensional quantities: N_1, A, and β. This relationship is illustrated graphically in Fig. 7.

Figures 6 and 7 describe the quantitative behavior of a plasma-beam system under various specific conditions. Assume that the stationary plasma is known, i.e., the value of the parameter A is given. One selects in Fig. 6 a graph corresponding to this value and representing a relationship between X and β. Since Ω_i is also known, one can ascertain from this graph the frequencies $\tilde{\omega}$ of resonant waves which may be excited by a beam of any given velocity β. The corresponding value of N_1 may be obtained from the graph in Fig. 7 which corresponds to the same value of A. When N_1 is known, the permissible value of σ can be determined by taking into account the inequality $\sigma \ll 1$ and expressions (1-30) and (1-69). The relative rate of growth of the wave can then be determined from (1-67).

The quantity N_1 may also be expressed explicitly as a function of X and β. Thus, substituting A^2 as given by (1-63) in (1-68), one obtains

$$N_1=\left|\frac{(1-\beta^2)^{3/2}[(1-\beta^2)^{1/2}-2X\alpha](X+1)^2}{\alpha^2X^3[2\alpha^2\beta^2X^2(X+1)+(1-\beta^2)^{1/2}[(1-\beta^2)^{1/2}-2X\alpha](X+2)]}\right|^{1/2}. \tag{1-74}$$

A simplified form of relationship (1-68) will be derived for each of the two extreme cases $A\gg 1$ (i.e., $\omega_i \gg \Omega_i$) and $A \ll 1$ (i.e., $\omega_i \ll \Omega_i$). When $A \gg 1$, the parameter N_1 can be expressed as follows:

$$N_1=\left|\frac{(1-\beta^2)(X+1)^2}{\alpha^2X^3(X+2)}\right|^{1/2}. \tag{1-75}$$

When $|\bar{\omega}| \ll \Omega_i$ (i.e., when $|X| \ll 1$), (1-68) yields the following expression:

$$N_1=|(1-\beta^2)/2\alpha^2X^3|^{1/2}. \tag{1-76}$$

It is shown in Sec. 3 of this investigation that for $|\bar{\omega}| \ll \Omega_i$ one has a hydromagnetic wave and, therefore, expression (1-76) is applicable to such a wave.

When $A \ll 1$, the parameter N_1 can be expressed as

$$N_1=|(1-\beta^2)A^2/2\alpha^2X^3|^{1/2}. \tag{1-77}$$

It is shown in Sec. 3 of this investigation that when $A \ll 1$, the frequency of the excited wave is close to the ion cyclotron frequency and expression (1-77) applies to such a case. When the excited wave is close to the ion cyclotron frequency, expression (1-77) yields the following form for the expression (1-69):

$$\sigma^{1/2} \ll \sqrt{2}\alpha/A\,(1-\beta^2)^{1/2}. \tag{1-78}$$

9. *Graphical Method of Bernstein and Trehan*

In a recent investigation, Bernstein and Trehan[2] explored various plasma-beam instabilities by means of a graphical procedure. Thus the quantity $y=\phi$ as defined by (1-4) was plotted as a function of ω (see Figs. 8, 9, and 10). In order to conform to the notation used by Bernstein and Trehan, the sign of the variable on each axis has been reversed.

There are several distinctive features which are different in the Bernstein and Trehan graphs from those shown in Figs. 4 and 5, which illustrate the behavior of functions $y=(F)_{\omega=\bar{\omega}}$ and $y=\psi(\bar{\omega}_-)$, respectively. These functions represent a stationary plasma, whereas in the Bernstein-Trehan representation the function $y=\phi$ represents the plasma-beam system. The independent variable in Figs. 4 and 5 represents the frequency $\bar{\omega}$ of resonant waves, whereas in the Bernstein-Trehan graphs the independent variable ω may not be in resonance with the beam. The Bernstein-Trehan representation is based on an assumption that k and β are fixed. On the other hand, in Fig. 4 an assumption was made that k is fixed and the variable quantities $\bar{\omega}$ and β are related to each other by the expression

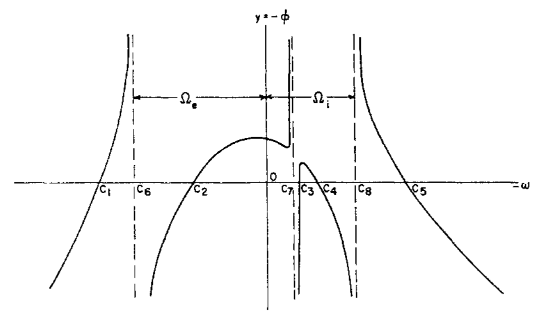

FIG. 8. Graph of $y=\phi$ for $\beta=\beta_1$; $OC_7=ck\beta_1-\Omega_e(1-\beta^2)^{1/2}$; $OC_3=ck\beta_1-\Omega_e(1-\beta^2)^{1/2}+\delta$.

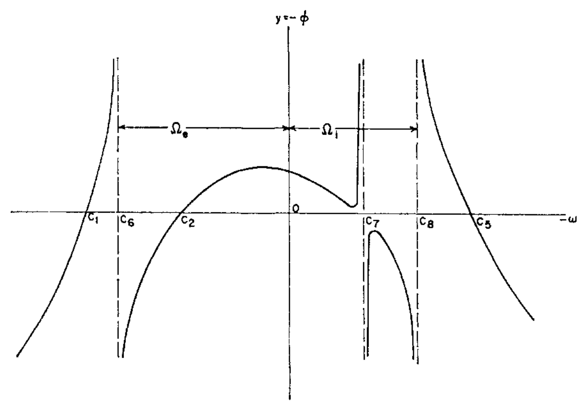

FIG. 9. Graph of $y=\phi$ for $\beta=\beta_2>\beta_1$; $OC_7=ck\beta_2-\Omega_e(1-\beta^2)^{1/2}$.

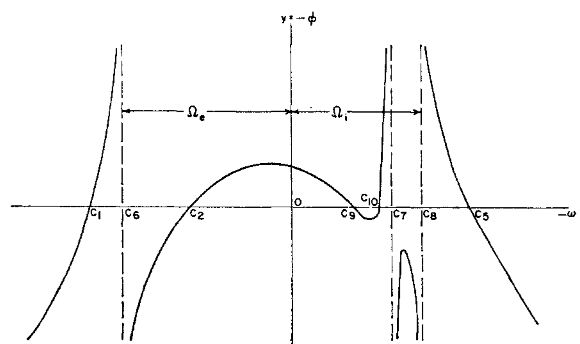

FIG. 10. Graph of $y=\phi$ for $\beta=\beta_3>\beta_2$; $OC_7=ck\beta_3-\Omega_e(1-\beta^2)^{1/2}$; $OC_{10}=ck\beta_3-\Omega_e(1-\beta^2)^{1/2}-\delta$.

$\tilde{\omega}=ck\beta+\Omega_e(1-\beta^2)^{1/2}$. In Fig. 5 an assumption was made that β is fixed and the variable quantities $\tilde{\omega}$ and k are related to each other by the same expression.

It has been shown by Bernstein and Trehan that by varying the velocity of the beam, one changes the character of the graphs in such a manner that for critical values of β some of the points of intersection of the curve $y=-\phi$ with axis of abscissas "disappear." The real roots of the dispersion equation $\phi=0$ are then replaced by complex roots which indicate an instability. The graphs of Figs. 8, 9, and 10 illustrate certain additional features of the Bernstein-Trehan representation. It is shown that if one increases the beam velocity β beyond the values considered by Bernstein and Trehan, some of the points of intersection may "reappear," and the system becomes stable again.

Assume that the value of β is given, i.e., $\beta=\beta_1$, and consider Fig. 8 representing the graph $y=-\phi$ corresponding to this value. There are five points of intersection of the graph $y=-\phi$ with the ω axis. These points are designated as $C_1\cdots C_5$, and represent all the roots of the dispersion equation (1-4). Since these roots are real, there is no instability.

A similar graph is given in Fig. 9. However, in this figure the beam velocity $\beta=\beta_2$ is larger than the beam velocity $\beta=\beta_1$ represented in Fig. 8. This increase introduced qualitative changes in the behavior of the graph $y=-\phi$. Thus in Fig. 9 the point C_7 representing the quantity $ck\beta_2-\Omega_e(1-\beta^2)^{1/2}$ has moved toward the point C_8 representing Ω_i. The portion of the curve contained between points C_7 and C_8 has been compressed and its maximum has become lower. There exists a limiting value of β above which this portion does not intersect the axis of abscissas. When this happens, the dispersion equation "loses" two real roots such as C_3 and C_4 and gains two complex conjugate roots. This is shown in Fig. 9. In this figure there are only three points of intersection of the curve $y=-\phi$ with the axis of abscissas. The two lost roots become complex and represent an instability.

By increasing the beam velocity again, the portion of the curve between the points C_7 and C_8 is compressed further and the maximum of this portion continues to go down. It is interesting to note, however, that for sufficiently small values of σ the instability disappears although there are no points of intersections within the compressed portion of C_7C_8. This is shown in Fig. 10 for which $\beta=\beta_3$ where $\beta_3>\beta_2$. The points of intersection which disappeared in the compressed portion as shown in Fig. 9 reappear in Fig. 10 in the portion of the curve between the points O and C_7. Consequently, the graph of Fig. 10 provides five real roots for ω, and there is no instability.

The "transition" from Fig. 8 to Fig. 9 is similar to the one illustrated by Bernstein and Trehan. It shows that for increasing values of the velocity β, an initially stable system may become unstable. One should observe, however, that if the velocity is increased further, one may obtain an additional "transition" from Fig. 9 to Fig. 10. In this additional transition the two points of intersection that previously vanished appear again, and, therefore, there is no instability. This additional transition is significant since it shows that the beam velocity associated with an instability is contained within a range having an upper and a lower bound.

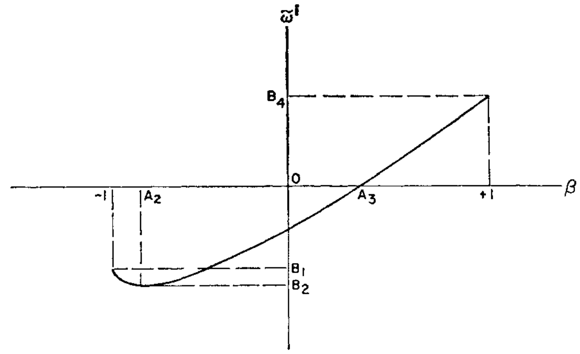

FIG. 11. Relationship between the angular frequency $\tilde{\omega}^i$ of a resonant wave and the linear velocity β of the ion beam.

2. PASSAGE OF AN ION BEAM THROUGH PLASMA

I. General Considerations

Consider now a charge equilibrated system in which the fraction σ_i of the ions are moving with velocity $\mathbf{v}=\boldsymbol{\beta}c$ through a stationary plasma immersed in a magnetic field $\mathbf{B}_0\|\boldsymbol{\beta}$. The stationary plasma consists of electrons and a fraction $1-\sigma_i$ of the ions. The assumptions made here are the same as in the case of an electron beam passing through plasma. It is assumed that the plasma is cold, the intensity of the beam is very small, and the wave resulting from the plasma-beam interaction is aligned along the direction of the magnetic field.

The dispersion equation may be represented as[13]

$$\phi_i\equiv\phi_i(\omega,k)=F-\frac{\sigma_i(1-\beta^2)^{1/2}\omega_i^2(\omega-ck\beta)}{\omega-ck\beta+\Omega_i(1-\beta^2)^{1/2}}, \quad (2\text{-}1)$$

where F is given by (1-5). It is assumed that $\sigma_i\ll 1$. As in the case of the electron beam $k>0$ represents an H_+ wave and $k<0$ represents an H_- wave.

It can be shown that the ion beam is capable of exciting only those waves which are in resonance with the ion beam. The frequency ω of the resonant wave can be expressed as

$$\omega=\tilde{\omega}^i+\delta, \quad (2\text{-}2)$$

where

$$\tilde{\omega}^i\approx ck\beta-\Omega_i(1-\beta^2)^{1/2}. \quad (2\text{-}3)$$

The relationship (2-3) is represented graphically in Fig. 11 for an H_+ wave.

The solution of the dispersion equation (2-1) can be written as

$$\delta=\left[-\frac{\sigma\omega_i^2\Omega_i(1-\beta^2)^{1/2}}{(\partial F/\partial\omega)_{\omega=\tilde{\omega}^i}}\right]^{1/2}, \quad (2\text{-}4)$$

subject to the conditions

$$|\delta|\ll|\tilde{\omega}^i|;\quad (F)_{\omega=\tilde{\omega}^i}\approx 0, \quad (2\text{-}5)$$

and also that a valid approximation is obtained by retaining two terms of the series expansion for $(F)_{\omega=\tilde{\omega}^i+\delta}$:

$$F=(F)_{\omega=\tilde{\omega}^i}+\delta(\partial F/\partial\omega)_{\omega=\tilde{\omega}^i}+\cdots. \quad (2\text{-}6)$$

Consider the waves which may be excited in a plasma by an ion beam of a known velocity β. Substituting in the dispersion equation $F=0$ the values for ω and k which correspond to resonant waves, i.e.,

$$\omega=\tilde{\omega}^i,\quad k=[\tilde{\omega}^i+\Omega_e(1-\beta^2)^{1/2}]/c\beta, \quad (2\text{-}7)$$

one obtains the following equation for waves excited by an ion beam:

$$\psi_i(\tilde{\omega}^i)=(\tilde{\omega}^i)^2-\frac{1}{\beta^2}[\tilde{\omega}^i+\Omega_i(1-\beta^2)^{1/2}]^2 -\frac{\omega_i^2\tilde{\omega}^i}{\tilde{\omega}^i+\Omega_i}-\frac{\omega_i^2\tilde{\omega}^i}{\alpha\tilde{\omega}^i-\Omega_i}=0. \quad (2\text{-}8)$$

Figure 12(a–c) shows three different forms of the graph of the function $y=\psi_i(\tilde{\omega}^i)$. These graphs correspond to three sets of values of β and A. The points of intersection of each graph with the axis of abscissas represent the roots of the equation $\psi_i(\tilde{\omega}^i)=0$. In order to determine which of these roots represents an instability, one should consider the expression (2-4) for δ and ascertain whether this quantity is real or complex. Substituting (1-5) in (2-4), one obtains

$$\delta^2=-\frac{\sigma\omega_i^2(1-\beta^2)}{2Y+A^2/(\alpha Y-1)^2-A^2/(Y+1)^2}, \quad (2\text{-}9)$$

where

$$Y=\tilde{\omega}^i/\Omega_i \quad\text{and}\quad A=\omega_i/\Omega_i. \quad (2\text{-}10)$$

It can be seen from expression (2-9) that δ^2 is negative and, consequently, there is an instability only if $\tilde{\omega}^i>0$. By applying arguments similar to those used for an electron beam, it can be shown that $\tilde{\omega}^i>0$ corresponds to a superluminous velocity of the ion beam both for H_+ and H_- waves.

The graph of Fig. 12(a) shows only one point of intersection with the axis of abscissas for which $\tilde{\omega}^i>0$. This point, labeled A_3, represents the only wave that may be excited by the ion beam. A similar situation is represented in Fig. 12(b) in which the ion beam produces a single instability labeled A_9. It should be noted, however, that there are values of the parameters A and β for which the graphical representation of $y=\psi_i(\tilde{\omega}^i)$ shows some distinctive qualitative changes. This is shown in Fig. 12(c) in which one obtains three points of intersection characterized by $\tilde{\omega}^i>0$. These points, labeled A_{13}, A_{14}, and A_{15}, represent three resonant

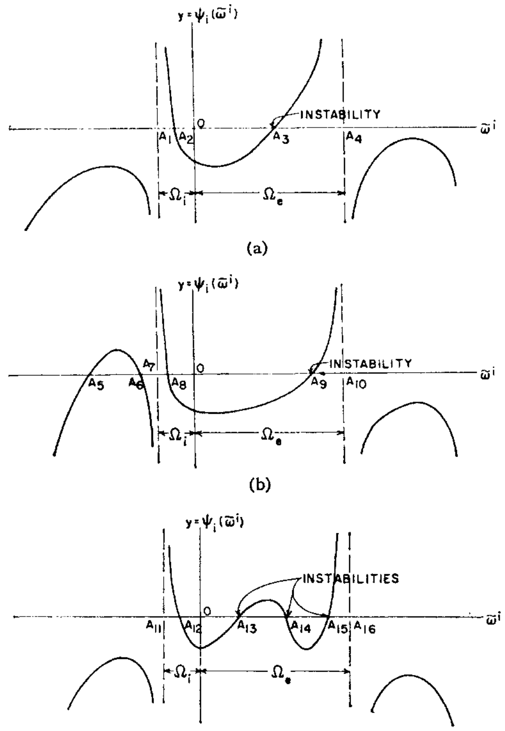

FIG. 12. (a)–(c) Graphical representation of $y=\psi_i(\tilde{\omega}^i)$ for three different values of β.

waves which may be excited by an ion beam. The occurrence of three resonant waves was pointed out by Ginzburg.[11]

It can be seen from Figs. 12(a), 12(b), and 12(c) that the frequencies $\tilde{\omega}^i$ characterizing the ion beam instabilities satisfy the following relationship:

$$0<\tilde{\omega}^i<\Omega_e=\Omega_i M_i/m=\Omega_i/\alpha. \quad (2\text{-}11)$$

Consequently, the frequency of an excited wave may extend from very low frequencies for which $\tilde{\omega}^i\ll\Omega_i$ up to the electron gyrofrequency. One can achieve a condition near resonance in which the wave frequency may approach the electron gyrofrequency from below and the difference $(\Omega_e-\tilde{\omega}^i)$ may be relatively small.

By applying a method similar to the one used for the electron beam, it can be shown that the excitation of H_+ wave occurs only if the beam moves with superluminous velocity in the direction of the magnetic field. The angular velocity of the H_+ wave has the same direction as the angular velocity of the perturbed stationary electrons in the plasma.

II. Frequencies of Excited Waves

Using $Y=\tilde{\omega}^i/\Omega_i$ and $A=\omega_i/\Omega_i$, Eq. (2-8) may be expressed as

$$Y^2-\frac{1}{\beta^2}[Y+(1-\beta^2)^{1/2}]-\frac{A^2Y^2(1+\alpha)}{(Y+1)(\alpha Y-1)}=0. \quad (2\text{-}12)$$

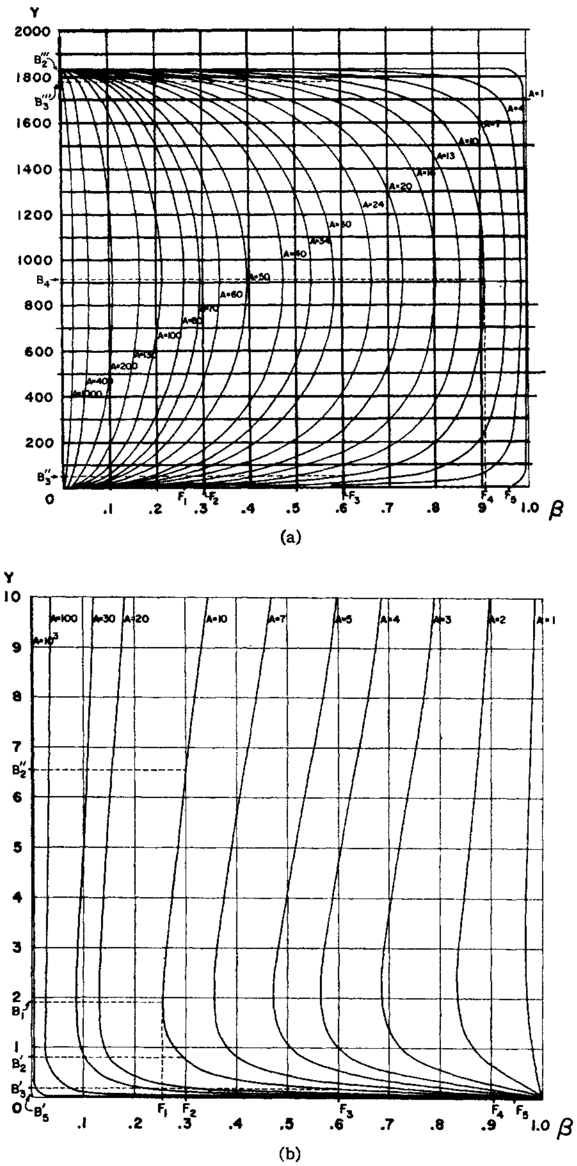

FIG. 13. (a) Graphical representation of the relationship between Y, A, and β (for an ion beam). (b) Enlarged portion of the graph of (a).

It can be seen that when $Y>0$ (i.e., when the velocity of the ion beam is superluminous), the equality (2.12) may be satisfied only if

$$0<Y<1/\alpha. \qquad (2\text{-}13)$$

This inequality is identical with the inequality (2-11) resulting from the graphical method of Figs. 12(a), 12(b), and 12(c).

The expression (2-12) describes a functional relationship between three nondimensional quantities Y, A, and β. This expression is analogous to the expression (1-58) obtained for an electron beam. Various graphs representing the behavior of the relationship (2-12) are shown in Figs. 13(a) and 13(b).

III. Rate of Growth

Neglecting in (2-9) α when compared to 1, the rate of growth for a wave excited by an ion beam can be expressed as

$$\mathrm{Im}\delta=\left|\frac{-\sigma_i\omega_i^2(1-\beta^2)(\alpha Y-1)^2(Y+1)^2}{2Y(\alpha Y-1)^2(Y+1)^2+A^2Y(Y+2)}\right|^{1/2}. \qquad (2\text{-}14)$$

The relative rate of growth $N=\mathrm{Im}\delta/\tilde{\omega}^i$ can be expressed in the form $N=\sigma_i^{1/2}N_1$ where

$$N_1=\left|\frac{-A^2(1-\beta^2)(\alpha Y-1)^2(Y+1)^2}{Y^3[2(\alpha Y-1)^2(Y+1)^2+A^2(Y+2)]}\right|^{1/2}. \qquad (2\text{-}15)$$

The assumption $N\ll 1$ places the restriction $\sigma_i^{1/2}\ll 1/N_1$ on the permissible density of the beam.

Combining Eqs. (2-12) and (2-15), the term Y can be eliminated, yielding an expression which describes a functional relationship between the nondimensional quantities N_1, A, and β. This relationship is analogous to the expression (1-70) obtained for an electron beam and is illustrated graphically in Fig. 14.

The behavior of instabilities under various conditions can be ascertained by means of Figs. 13(a), 13(b), and 14. Thus, if the parameter A of the plasma is known, one obtains from Fig. 13(a) or (b) a graph representing a relationship between Y and β. Since Ω_i is known, one can ascertain from this graph the frequencies of excited resonant waves and the corresponding velocities of the ion beam. By means of Fig. 14 one obtains the values of N_1 corresponding to these frequencies and velocities. Taking into account $\sigma_i\ll 1$, $\sigma_i\ll 1/N_1$, and (2-6), the permissible values of σ_i and the relative rates of growth can be determined.

If $A\gg 1$, the term N_1 can be expressed in the simplified form:

$$N_1=\left|\frac{-(1-\beta^2)(\alpha Y-1)^2(Y+1)^2}{Y^3(Y+2)}\right|^{1/2}. \qquad (2\text{-}16)$$

3. COMPARISON OF THE EXCITATION MECHANISMS PRODUCED BY ELECTRON AND ION BEAMS

I. General Considerations

Both an electron beam and an ion beam are capable of exciting electromagnetic waves when the following three conditions are satisfied: (a) the waves move in the same direction as the beam, (b) the beam moves with superluminous velocity, and (c) the waves are "resonant" with the beam.

Resonant waves excited by an electron beam are different from those excited by an ion beam. For an electron beam the frequency of the excited wave is in resonance with the gyrofrequency of the electrons in

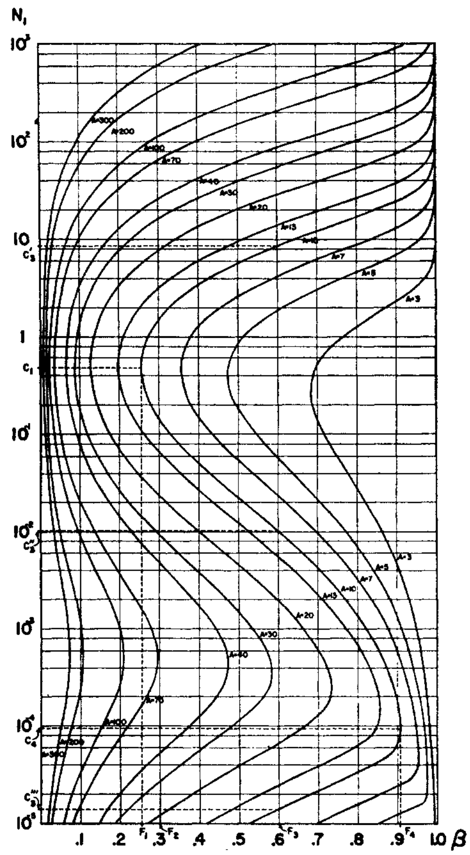

FIG. 14. Graphical representation of the relationship between N_1, A, and β (for an ion beam).

the beam, whereas for an ion beam the frequency of the excited wave is in resonance with the gyrofrequency of the ions in the beam.

Both H_+ and H_- waves may be excited by a beam. A beam of electrons moving in the direction of the magnetic field is capable of exciting H_- waves only, whereas only H_+ waves are excited when the beam moves in the direction opposite to the magnetic field.

Due to the resonance effect, both H_+ and H_- waves excited by an electron beam rotate in the same direction and with the same angular frequency as the gyroelectrons of the beam. There occurs, however, an "anomalous effect" represented by a reversal in the direction of rotation caused by the superluminous velocity of the beam. Consequently, an excited H_+ or H_- wave has a circular motion in the same direction as the gyrofrequency of the stationary ions. Furthermore, the frequency of such a wave is contained within the range $-\Omega_i < \bar{\omega} < 0$.

An ion beam moving in the direction of the magnetic field is capable of exciting H_+ waves only, whereas only H_- waves may be excited when the ion beam moves in the direction opposite to the magnetic field. The excited H_+ and H_- waves rotate inthe same direction as the perturbed electrons in the stationary plasma. The frequency of a wave excited by an ion beam is always below the frequency of the perturbed stationary electrons. Thus the ion beam may excite waves having frequencies $\bar{\omega}^i$ within the range $0 < \bar{\omega}^i < \Omega_e$. This range is considerably wider than the range of frequencies $-\Omega_i < \bar{\omega} < 0$ excited by an electron beam.

It may be of interest to point out that an ion beam is capable of exciting a wave having frequency $\bar{\omega}^i$ which is numerically equal to the ion gyrofrequency Ω_i. However, this inequality does not have any particular physical significance since it does not represent an ionic resonance. While the frequencies of the excited wave and of the perturbed ion are the same, the directions of rotation are opposite to each other.

There are significant qualitative differences in the behavior of the instabilities produced by an electron beam and those produced by an ion beam. These differences may be readily observed from Figs. 6, and 13(a) and (b). Thus, it is seen from Figs. 13(a) and (b) that for certain values of the parameters A and β an ion beam may excite simultaneously three waves. This effect has already been noted by M. A. Ginzburg.[11] However, it can be seen from Fig. 6 that an electron beam may excite only one wave for a given set of values of A and β.

The quantity $A = \omega_i/\Omega_i = (4\pi n M_i)^{1/2} c/B_0$ which depends upon the strength of the magnetic field and the density of the medium is very important in determining the behavior of a plasma. A very useful classification of various types of plasma was recently introduced by Denisse and Delcroix.[12] Thus a plasma is "very rarefied" when $A < .07$ and "rarefied" when $.07 < A < 1$. Both rarefied and very rarefied plasma occurs in evacuated vessels having pressure of the order of 10^{-5} mm of Hg in the presence of a very strong magnetic field (cyclotrons, vacuum gauges, etc.). For the ionosphere, one has $A \sim 1.5 \times 10^2$, and in such a case the plasma is "dense." The dense plasma is characterized by $A > 50$ and therefore there are other examples of dense plasma such as thermonuclear discharges, interstellar clouds, etc. A plasma of small density is characterized by $1 < A < 50$. Various air-discharges and solar corona may be represented by a dense plasma or by a plasma of small density.

Figure 15 illustrates the typical behavior of various plasmas as classified above. The effect of electron beams on various types of plasma is based on Fig. 6, whereas the effect of ion beams is based on Figs. 13(a) and 13(b).

II. Excitation of Waves by an Electron Beam

Consider now the graphs of Figs. 6 and 7. Figure 6 shows the frequencies which may be excited for various values of A and β. A separate curve has been plotted of X as a function of β for each of the several values of A. The values of A extend from $A = 10^2$ to $A = 10^6$. For each A the appropriate curve shows a one to one correspondence between X and β. Thus a beam characterized

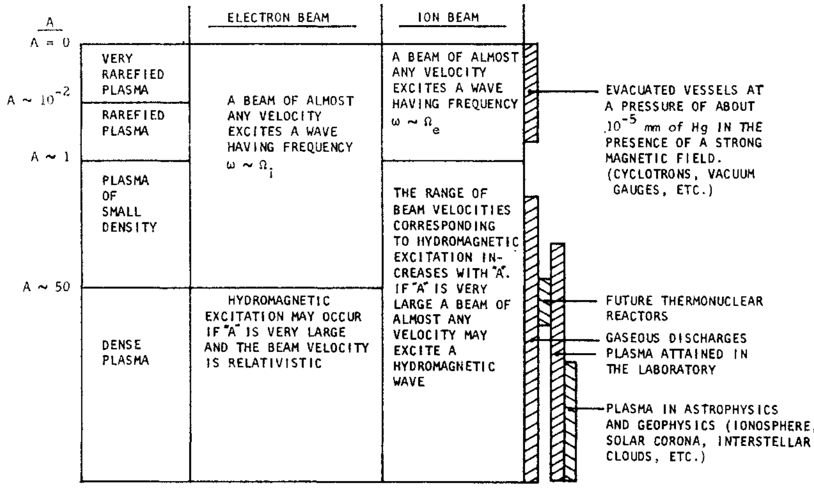

FIG. 15. Typical behavior for various plasmas.

by a given value of β is capable of exciting only one wave determined by the corresponding value of X.

The values $X \sim 1$ and $X \ll 1$ are of particular interest and the conditions under which frequencies represented by these values can be excited shall be determined. When $X \sim 1$, one has $\bar{\omega} \sim \Omega_i$ and the excited wave has a frequency which is very close to the ion gyrofrequency (but somewhat below the ion gyrofrequency). When $X \ll 1$, one has $\bar{\omega} \ll \Omega_i$. One may obtain then a hydromagnetic wave if certain other conditions are also satisfied. The criterion for the occurrence of a hydromagnetic wave in an undisturbed stationary plasma may be obtained from the dispersion equation $F=0$. Using the inequality $\omega = \bar{\omega} \ll \Omega_i$, one obtains[19]

$$\bar{\omega}^2 = c^2k^2/(1+c^2/V_A{}^2), \tag{2-17}$$

where

$$V_A = B_0/(4\pi n M_i)^{1/2}. \tag{2-18}$$

Applying the relationships (1-1) and (1-2), and using the term $A = \omega_i/\Omega_i$, (2-17) can be expressed as

$$\bar{\omega}^2/k^2 = c^2/(1+A^2). \tag{2-19}$$

Expression (2-19) represents the dispersion equation for a low-frequency wave which may be propagated in an undisturbed plasma described by the dispersion equation (1-5). Of particular interest are two limiting forms of this wave. Thus assuming $A \gg 1$, one obtains from (2-19)

$$\bar{\omega}/k \sim c/A = V_A. \tag{2-20}$$

Expression (2-20) represents a low-frequency wave moving with Alfvén velocity V_A. On the other hand, when $A \ll 1$, one obtains

$$\bar{\omega}/k = c. \tag{2-21}$$

The above expression represents a low-frequency wave propagated with the velocity of light.

[19] See, for instance, I. B. Bernstein and K. Trehan, Nucl. Fusion 1, 3 (1960).

One can ascertain now which one of the two low-frequency waves may be excited by an electron beam—the slowly moving hydromagnetic wave represented by (2-20) or the wave (2-21) moving with the velocity of light.

Consider first the wave represented by (2-21). A necessary condition for the occurrence of such a wave is given by the inequality $A \ll 1$. It is noted that the curves plotted in Fig. 6 correspond to various values of A for which this inequality is not satisfied since these curves correspond to $A \gg 1$. The curves satisfying the inequality $A \ll 1$ cannot be conveniently represented on the scale of Fig. 6. All of these curves would be approximated very closely by a straight line parallel to the β axis and corresponding to $X \sim 1$. Thus the graphs for $A \ll 1$ represent loci of points for which $\bar{\omega} \sim \Omega_i$. Therefore, for $A \ll 1$ the beam is capable of exciting only waves having frequencies close to the ion gyrofrequency and one may assume that there are no low-frequency waves excited by the beam. Consequently, excited waves of the type (2-21) are nonexistent in the plasma-beam system.

One can, however, excite a hydromagnetic wave since each of the graphs plotted in Fig. 6 corresponds to $A \gg 1$. It is noted that for each A there is a different range of values of β which is needed for the excitation of a hydromagnetic wave. Thus for $A = 10^6$ one may excite a hydromagnetic wave when the velocity of the beam occupies a relatively wide range extending from very low values of β up to β approaching 1. Thus, assuming that the excited hydromagnetic wave corresponds to $X = 0.01$, one obtains for β a range $0.17 < \beta < 1$. If A is smaller, the corresponding range for β is considerably shortened and the low velocity region is eliminated. Thus when $A = 10^5$ one has $0.87 < \beta < 1$ and for a somewhat smaller value of A such as $A = 5 \times 10^4$, one may excite a hydromagnetic wave (corresponding to $X = 0.01$) if the beam has a velocity in the relativistic range ($\beta \sim 0.99$).

Consider now the conditions which are necessary for exciting waves close to the ion gyrofrequency. These conditions are strongly dependent on the parameter A. The case of $A \ll 1$ was discussed above, and it has been pointed out that the frequency $\bar{\omega} \sim \Omega_i$ would be excited for nearly all values of β. This situation remains substantially the same when $A \sim 1$ and even when A is considerably larger than 1. One can assume from Fig. 6 that for any value of A comprised within the range $1 < A < 10$ the waves excited by the beam have frequency $\bar{\omega} \sim \Omega_i$ for all nonrelativistic values of β. This situation changes rapidly, however, for increasing values of A. Assume that the excited wave in the neighborhood of the ion gyrofrequency satisfies the relationship $X = 0.99$. Thus when $A = 10^3$, the beam may excite the wave if its velocity is comprised within the range $\beta < 0.19$. When A assumes still larger values, i.e., for $A = 10^4$ one may excite a wave corresponding to $X = 0.99$ only if the velocity of the beam is extremely small ($\beta < 0.02$).

Figure 7 together with Eq. (1-69) shows the relationship between the relative growth rate N and various values of A and β for a sufficiently small value of σ. As in Fig. 6, a separate curve has been plotted for N_1 [as expressed by the equation (1-70)] as a function of β for several values of A. An examination of Fig. 7 reveals at once that the relative growth rate increases with increasing values of A for a fixed value of β and increases with increasing values of β if A is fixed.

III. Excitation of Waves by an Ion Beam

Consider now the graphs of Figs. 13(a), (b), and 14. Figure 13(a) shows the frequencies which can be excited for various values of A and β. A separate curve has been plotted for Y as a function of β for each of several values of A. The value $Y = 1/\alpha = 1837$ corresponds to a resonance between the frequency $\bar{\omega}^i$ of the excited wave and the electron gyrofrequency Ω_e. Figure 13(b) has been drawn in order to show more clearly the behavior of the curves in Fig. 13(a) for small frequencies.

The set of curves shown in Figs. 13(a) and (b) correspond to the values of A from $A = 1$ to $A = 10^3$. These curves have two common points. One of these is represented by $\beta = 0$, $Y = 1/\alpha$, and the other by $\beta = 1$, $Y = 0$.

In order to point out the qualitative differences between the curves in Figs. 13(a) and (b) and those of Fig. 6, one can choose a particular numerical value for A and examine the relationship between Y and β. Assume, for instance, that $A = 10$ and consider the corresponding graph shown in Figs. 13(a) and (b). When β is very small, the corresponding values of Y shown in this graph are in the neighborhood of $Y = 1/\alpha$. Therefore, for $A = 10$, a very slowly moving ion beam excites frequencies which are close to the electron gyrofrequency. Consider now the values of β within the range $0 < \beta < OF_1$, where $OF_1 \sim 0.25$. There is a one to one relationship between β and Y within this range, i.e., for each value of β there is only one frequency $\bar{\omega}^i$ excited by the beam. One can also note a very slight decrease in Y (or $\bar{\omega}^i$) for increasing values of β. Therefore, it can be stated that within the range $0 < \beta < OF_1$, the beam excites only the waves for which $\bar{\omega}^i \sim \Omega_e$. When β reaches the value $\beta = OF_1$ and, furthermore, when β exceeds the value $\beta = OF_1$, a significant qualitative change can be observed in the behavior of the plasma-beam system. Thus when $\beta = OF_1$, the beam is capable of exciting simultaneously two different waves having two different frequencies. One of these has a value $\bar{\omega}^i \sim \Omega_e$ and the other a value $\bar{\omega} = OB_1$. When $\beta > OF_1$, the beam is capable of exciting simultaneously three different waves having three different frequencies. One of these frequencies is in the neighborhood of the electron gyrofrequency and the other two are generally considerably below the electron gyrofrequency. If the other condition is satisfied, i.e., $A \gg 1$, one obtains one or sometimes two hydromagnetic waves. Thus, when $\beta = OF_2 = 0.3$, the two low-frequency hydromagnetic waves are represented by $Y = OB_2'$ and by $Y = OB_2''$, and the high-frequency waves are represented by $Y = OB_2'''$. If a larger value of β such as $\beta = OF_3 = 0.6$ is considered, then one of the two low frequencies has an increased value and the other has a decreased value. These correspond to $Y = OB_3'$ and $Y = OB_3''$. The three excited waves occur simultaneously for all values of β comprised within the range $OF_1 < \beta < OF_4$ where $OF_4 \sim 0.91$. For β exceeding the value $\beta = OF_4$, another qualitative change in the behavior of the plasma-beam system is observed. Instead of three excited waves there is only one, and the excited wave is hydromagnetic. Thus for $\beta = OF_5$, the frequency of the hydromagnetic wave is represented by $Y = OB_5'$.

Similar considerations can be applied to other graphs corresponding to other values of A. Each of these graphs is characterized by two threshold values for the beam velocity β: the lower threshold β_l and the upper threshold β_u. Thus for $A = 10$, one has $\beta_l = OF_1$ and $\beta_u = OF_4$. These threshold values define three different ranges for the beam velocity β: the lower, the intermediate, and the upper range. The lower range corresponds to $\beta < \beta_l$. In this range the ion beam excites one wave only. This wave has a relatively high frequency which is often close to the electron gyrofrequency. In the intermediate range covering the values $\beta_l < \beta < \beta_u$, the beam excites three waves. In the upper range for which $\beta > \beta_u$ the beam excites a single wave which is hydromagnetic.

Figure 14 represents the graph of N_1 [as given by (2-15)] as a function of β for each of several fixed values of A. Figure 14 corresponds to an ion beam and is analogous to Fig. 7 corresponding to an electron beam.

It has been previously noted that for a given A, the graphs of Fig. 13(a) and (b) may yield three values of Y for a single beam velocity β. Each of these values of Y represents an excited frequency which is characterized by the corresponding value of N_1. Therefore, a similar situation is obtained in the representation of Fig. 14, i.e., the graph of Fig. 14 corresponding to the given value of A yields three values of N_1 for the same beam velocity. Therefore, a one to one correspondence needs to be established between the three values of Y obtained from Figs. 13(a) and (b) and the three values of N_1 obtained from Fig. 14 (assuming that A and β are the same). A comparison of Fig. 14 with Figs. 13(a) and (b) shows that points toward the top of the curves in Fig. 14 represent values of N_1 which correspond to points toward the bottom of the curves in Figs. 13(a) and (b).

In order to illustrate the above relationship, consider again the value $A=10$. If β is less than OF_1, then N_1 is less than 10^{-5}. This value for N_1 is relatively small and it has not been represented in Fig. 14. If $\beta=OF_1$, then the frequency $Y=OB_1$ in Fig. 13(b) would yield in Fig. 14 the value OC_1 for N_1. For $\beta=OF_3$, the values OC_3', OC_3'', and OC_3''' for N_1 correspond, respectively, to the frequencies OB_3', OB_3'', and OB_3'''. As β increases toward the value OF_4, the two values OC_3'' and OC_3''' approach the value OC_4 which is the value of N_1 corresponding to the frequency OB_4 on Fig. 13(a).

Thus it can be seen that N_1 increases as the frequency Y represented in Figs. 13(a) and (b) decreases.

It should be noted, however, that higher values of N_1 place more stringent conditions on the permissible values of σ since the condition $\sigma^{1/2} \ll 1/N_1$ must be satisfied.

IV. Graphical Representation of the Dispersion Equation

According to Sturrock,[20] one can ascertain from the graphical representation of the dispersion equation in the ω-k plane whether an instability is convective or nonconvective. In a convective instability a disturbance increases as it is carried along the system, and it remains finite at each point. In a nonconvective instability a disturbance which originated in a limited region of space at any instance of time grows indefinitely for $t\rightarrow\infty$ in this region.

In investigating the plasma-beam instabilities it may be useful to represent the dispersion equation of a stationary plasma in the form of a "ω-k" diagram and then ascertain how the character of such a diagram is modified by the presence of a beam. Consider in that connection the "ω-k" diagram of Fig. 16. This diagram represents the dispersion equation $F(\omega,k)=0$ where F is given by (1-5), and it describes, therefore, the behavior of the medium in the absence of a beam. One can observe that for any real value of k there are four real values of ω which satisfy the equation $F(\omega,k)=0$.

Figure 17 represents the "ω-k" diagram resulting from an interaction of an electron beam with a stationary plasma, and Figs. 18(a), (b), and (c) illustrate the interaction of an ion beam with a stationary plasma. It is noted again that Figs. 16 through 18(c) have not been drawn to scale in order to show more clearly the qualitative behavior of the functions represented by the corresponding graphs.

1. *Instability Produced by an Electron Beam*

The presence of an electron beam gives rise to an additional term in the dispersion equation. This equation is given by $\phi(\omega,k)=0$ in (1-4). The term

$$\frac{\sigma\omega_e^2(1-\beta^2)^{1/2}(\omega-ck\beta)}{\omega-ck\beta-\Omega_e(1-\beta^2)^{1/2}} \tag{4-1}$$

results from the presence of the beam. If σ is very small, the only portions of the graph in Fig. 15 which will be appreciably affected are in the neighborhood of the line $\omega=ck\beta+\Omega_e(1-\beta^2)^{1/2}$. The "$\omega$-$k$" diagram representing the dispersion equation $\phi(\omega,k)=0$ is given in Fig. 16.

The rectangle labeled R in the third quadrant of Fig. 16 represents a region of convective instability

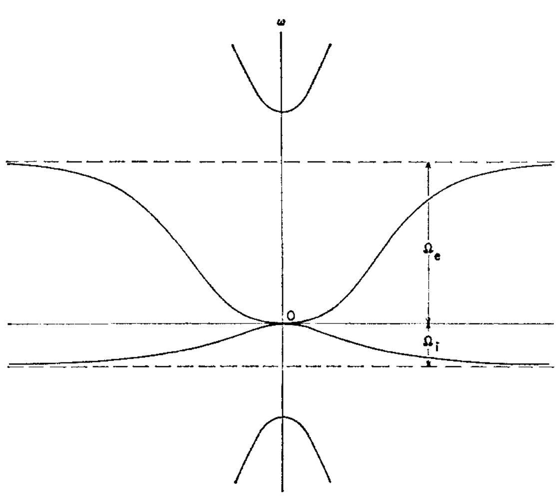

FIG. 16. "ω-k" diagram for a stationary plasma.

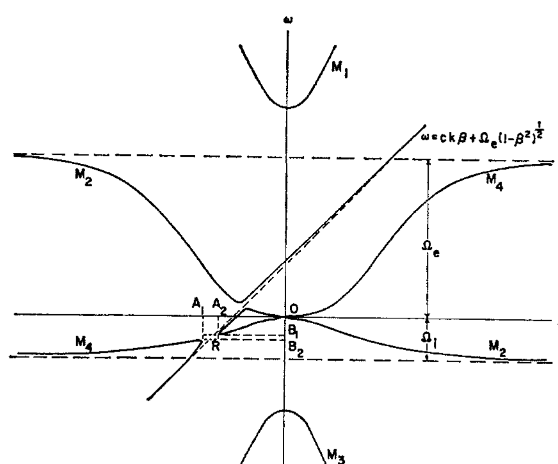

FIG. 17. "ω-k" diagram for an electron beam interacting with a stationary plasma.

[20] P. A. Sturrock, Phys. Rev. **112**, 1488 (1958). See also additional remarks in Jacob Neufeld and Harvel Wright, Phys. Rev. **124**, 3-4 (1961).

(i.e., a region in which ω is complex for real k and k is complex for real ω).

The frequency range B_1B_2 represented in this region depends on the parameters of the system. However, in agreement with previous discussions, the frequencies $\bar{\omega}$ comprised in this range must be negative and satisfy the relationship $|\bar{\omega}| < \Omega_i$.

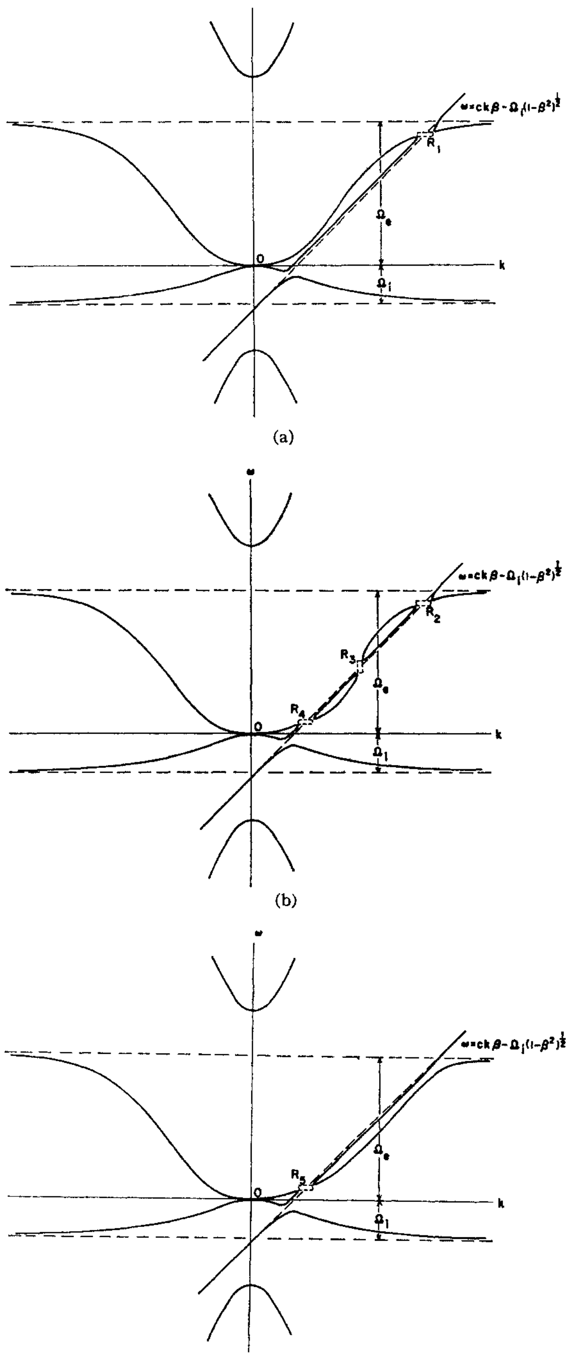

FIG. 18. (a) "ω-k" diagram for an ion beam interacting with a stationary plasma. (The velocity of the beam is in the lower range.) (b) "ω-k" diagram for an ion beam interacting with a stationary plasma. (The velocity of the beam is in the intermediate range.) (c) "ω-k" diagram for an ion beam interacting with a stationary plasma. (The velocity of the beam is in the upper range.)

2. *Instabilities Produced by an Ion Beam*

The presence of an ion beam introduces a term in the dispersion equation $F(\omega,k)=0$ which is different from the one introduced by an electron beam. The dispersion equation in this case becomes $\phi_i(\omega,k)=0$, as given by (2-1) where the term

$$\frac{\sigma_i\omega_i^2(1-\beta^2)^{1/2}(\omega-ck\beta)}{\omega-ck\beta+\Omega_i(1-\beta^2)^{1/2}} \tag{4-2}$$

is introduced by the beam. Again, if σ is very small, the graph in Fig. 15 will be appreciably disturbed only in the neighborhood of the line $\omega=ck\beta-\Omega_i(1-\beta^2)^{1/2}$. The diagrams for this case are given in Figs. 18(a), (b), and (c). The instabilities occur in the rectangular regions in the first quadrant labeled R_1 through R_5. It can be readily seen that these instabilities are convective.

Comparing Figs. 13(a) and (b) with Figs. 18(a), (b), and (c), one can see more clearly the relationship between the "ω-k" diagrams and the previous discussions. Consider in that connection the instabilities produced by an ion beam in a stationary plasma for which $A=10$. Therefore, one refers to the appropriate graph in Figs. 13(a) and (b) and compares the relationship between β and Y as indicated by this graph with the instabilities shown in Figs. 18(a), (b), and (c). Figure 18(a) represents an instability produced by an ion beam when its velocity β is in the lower range, i.e., when $\beta < OF_1$. In such case the instability located in the neighborhood of the region R_1 has a frequency which is very close to the electron gyrofrequency. Assume now that β increases and enters into the intermediate range for which $OF_1 < \beta < OF_4$. In such case the slope of the line $\omega=ck\beta-\Omega_i(1-\beta^2)^{1/2}$ increases and at the same time the line moves upward [$(1-\beta^2)^{1/2}$ decreases so that $-\Omega_i(1-\beta^2)^{1/2}$ increases]. The "ω-k" diagram is then represented by Fig. 18(b) which shows three instabilities in the neighborhood of the regions R_2, R_3, and R_4. One of these instabilities, in the neighborhood of the region R_2, has a frequency which is close to the electron gyrofrequency. Another instability in the neighborhood of the region R_4 represents a hydromagnetic wave. When the velocity of the ion beam increases further and enters the upper range for which $\beta > OF_4$, one obtains a diagram as shown in Fig. 18(c). In this range there is only one instability located in the region R_5. This instability represents a growing hydromagnetic wave.

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